FIXED POINT THEOREMS FOR MAPPINGS
WITH A CONTRACTIVE ITERATE AT A POINT

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ABSTRACT. Let \((X,d)\) be a complete metric space, \(T: X \rightarrow X\), and \(\alpha: [0, \infty)^5 \rightarrow [0, \infty)\) be nondecreasing with respect to each variable. Suppose that for the function \(\gamma(t) = \alpha(t,t,t,2t,2t)\), the sequence of iterates \(\gamma^n\) tends to 0 in \([0, \infty)\) and \(\lim_{t \rightarrow \infty} (t - \gamma(t)) = \infty\). Furthermore, suppose that for each \(x \in X\) there exists a positive integer \(n = n(x)\) such that for all \(y \in X\),

\[
d(T^n x, T^n y) < \alpha(d(x,T^n x),d(x,T^n y),d(x,y),d(T^n x,y),d(T^n y,y)).
\]

Under these assumptions our main result states that \(T\) has a unique fixed point. This generalizes an earlier result of V. M. Sehgal and some recent results of L. Khazanchi and K. Iseki.

1. For a function \(\gamma: [0, \infty) \rightarrow [0, \infty)\) denote by \(\gamma^n\), \(n = 0, 1, \ldots\), the \(n\)th iterate of \(\gamma\). Before stating the main result we prove the following.

**Lemma.** Suppose that \(\gamma: [0, \infty) \rightarrow [0, \infty)\) is nondecreasing. Then for every \(t > 0\), \(\lim_{n \rightarrow \infty} \gamma^n(t) = 0\) implies \(\gamma(t) < t\).

**Proof.** Suppose that for some \(t_0 > 0\) we have \(\gamma(t_0) > t_0\). Hence, by the monotonicity of \(\gamma\), \(\gamma^n(t_0) > t_0\) for \(n = 1, 2, \ldots\). This proves the lemma.

**Remark 1.** Note that for every right continuous function \(\gamma: [0, \infty) \rightarrow [0, \infty)\) such that \(\gamma(t) < t\) for \(t > 0\), \(\lim_{n \rightarrow \infty} \gamma^n(t) = 0\).

**Theorem 1.** Let \((X,d)\) be a complete metric space, \(T: X \rightarrow X\), \(\alpha: [0, \infty)^5 \rightarrow [0, \infty),\) and let \(\gamma(t) = \alpha(t,t,t,2t,2t)\) for \(t \geq 0\).

Suppose that

1°. \(\alpha\) is nondecreasing with respect to each variable,

2°. \(\lim_{t \rightarrow \infty} (t - \gamma(t)) = \infty\),

3°. \(\lim_{n \rightarrow \infty} \gamma^n(t) = 0, t > 0\),

4°. for every \(x \in X\), there exists a positive integer \(n = n(x)\) such that for all \(y \in X\),

\[
d(T^n x, T^n y) < \alpha(d(x,T^n x),d(x,T^n y),d(x,y),d(T^n x,y),d(T^n y,y)).
\]

Then \(T\) has a unique fixed point \(a \in X\) and for each \(x \in X\), \(\lim_{k \rightarrow \infty} T^k x = a\).

**Proof.** First we shall show that for every \(x \in X\), the orbit \(\{T^i x\}_{i=0}^\infty\) is bounded.

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To prove this assertion we fix an \( x \in X \), an integer \( s \), \( 0 \leq s < n = n(x) \), and we put
\[
 u_k = d(x, T^{kn+s}x), \quad k = 0, 1, \ldots, \\
 h = \max(u_0, d(x, T^n x)).
\]
By \( 2^\circ \) there is a \( c, c > h \), such that
\[
 t - y(t) > h, \quad t > c.
\]
From the choice of \( c \) we have \( u_0 < c \). Suppose that there exists a positive integer \( j \) such that \( u_j > c \). Evidently, we may assume that \( u_i < c \) for \( i < j \). Hence, by the triangle inequality,
\[
 d(T^n x, T^{(j-1)n+s}x) \leq d(x, T^n x) + u_{j-1} < 2u_j, \\
 d(T^{jn+s} x, T^{(j-1)n+s}x) \leq u_j + u_{j-1} < 2u_j.
\]
Now, using \( 4^\circ \) and \( 1^\circ \), we get
\[
 u_j = d(x, T^{jn+s}x) \leq d(T^n x, T^n T^{(j-1)n+s}x) + d(x, T^n x) \\
 \leq \alpha(u_j, u_j, u_j, 2u_j, 2u_j) + h = \gamma(u_j) + h,
\]
i.e. \( u_j - \gamma(u_j) < h \) which together with \( u_j > c \) contradicts to the choice of \( c \). Therefore \( u_j < c \) for \( j = 0, 1, \ldots \), and, consequently, the orbit \( \{T^i x\}_{i=0}^\infty \) is bounded.

Take an \( x_0 \in X \) and put \( n_0 = n(x_0) \). Define \( \{x_k\} \) as follows
\[
 x_{k+1} = T^n x_k, \quad n_k = n(x_k), \quad k = 0, 1, \ldots
\]
Evidently, \( \{x_k\} \) is a subsequence of the orbit \( \{T^i x_0\}_{i=0}^\infty \). We shall prove that \( \{x_k\} \) is a Cauchy sequence.

Let \( k \) and \( i \) be positive integers. From (1) we have
\[
 x_{k+i} = T^{n_{k+i-1} + \cdots + n_k} x_k.
\]
Denoting \( s_0 = n_{k+i-1} + \cdots + n_k \), we can write
\[
 d(x_k, x_{k+i}) = d(x_k, T^{s_0} x_k).
\]
For the simplicity of the notations we put \( t_i = d(x_{k-1}, T^i x_{k-1}) \). Denote by \( s_1 \) that of the numbers \( s_0, s_{k-1}, s_0 + n_{k-1} \) for which \( t_{s_1} \) has the greatest value. Thus, by the triangle inequality, we have
\[
 d(T^{n_{k-1} x_{k-1}}, T^{s_0} x_{k-1}) \leq t_{n_{k-1}} + t_{s_0} \leq 2t_{s_1}, \\
 d(T^{n_{k-1} + s_0} x_{k-1}, T^{s_0} x_{k-1}) \leq t_{n_{k-1} + s_0} + t_{s_0} \leq 2t_{s_1}.
\]
Now \( 4^\circ \) and \( 1^\circ \) imply
\[ d(x_k, T^{x_k}T^k) = d(T^{x_k}T^k, T^{x_k}T^k) \]
\[ \leq \alpha(t_1, t_2, t_1, 2t_1, 2t_1) = \gamma(t_1), \]
i.e.,
\[ d(x_k, T^{x_k}T^k) \leq \gamma(d(x_k, T^{x_k}T^k)). \]
Repeating this procedure, we can find positive integers \( s_j, j = 1, \ldots, k - 1, \)
such that
\[ d(x_k, T^{x_k}T^k) \leq \gamma(d(x_k, x_k)). \]
Hence, since \( \gamma \) is nondecreasing, we obtain
\[ d(x_k, x_k) \leq \gamma^k(M), \]
where \( M \) denotes the diameter of the orbit \( \{T^i x_0\}_{i=0}^\infty \). By 3°, \( \lim_{k \to \infty} \gamma^k(M) = 0 \). This proves that \( \{x_k\} \) is a Cauchy sequence.

By the completeness of \( X \) there is an \( a = \lim_{k \to \infty} x_k, a \in X \). We shall show that for \( n = n(a), T^n a = a \).

For an indirect proof suppose that \( \varepsilon = d(T^n a, a) > 0 \). Using the argument of the preceding part of the proof, we see that
\[ \lim_{k \to \infty} d(T^n a, x_k) = 0. \]
Therefore, by the Lemma, there exists a \( k_0 \) such that
\[ d(a, x_k) < \frac{1}{2}(\varepsilon - \gamma(\varepsilon)), \quad d(T^n a, x_k) < \frac{1}{2}(\varepsilon - \gamma(\varepsilon)), \quad k \geq k_0. \]
Hence we get
\[ \epsilon = d(T^n a, a) \leq d(T^n a, T^n x_k) + d(T^n x_k, x_k) + d(x_k, a) \]
\[ \leq \alpha(d(a, T^n a), d(a, T^n x_k), d(a, x_k), d(T^n a, x_k), d(T^n x_k, x_k)) \]
\[ + \frac{1}{2}(\varepsilon - \gamma(\varepsilon)). \]
Since \( d(a, T^n x_k) \leq d(a, x_k) + d(x_k, T^n x_k), \quad d(T^n a, x_k) \leq d(T^n a, a) + d(a, x_k), \) it follows that for \( k \geq k_0, \)
\[ d(a, T^n x_k) \leq \frac{1}{2}(\varepsilon - \gamma(\varepsilon)) < \epsilon, \quad d(T^n a, x_k) \leq 2\epsilon. \]
Therefore, by 1°,
\[ \epsilon \leq \alpha(\epsilon, \epsilon, \epsilon, 2\epsilon, 2\epsilon) + \frac{1}{2}(\varepsilon - \gamma(\varepsilon)) = \frac{1}{2}(\varepsilon + \gamma(\varepsilon)) < \epsilon, \]
which is a contradiction. Consequently, \( T^n a = a \).

Suppose that there is a point \( b \in X, b \neq a, \) such that \( T^n b = b \) with \( n = n(a) \). Then by 4° and the Lemma
\[ d(a, b) = d(T^n a, T^n b) \leq \alpha(0, d(a, b), d(a, b), d(a, b), 0) \]
\[ \leq \gamma(d(a, b)) < d(a, b). \]
This contradiction proves that \(a\) is a unique fixed point of \(T^n\).

Since \(T a = T^n T a\), just proved uniqueness yields \(T a = a\). Now it is trivial that \(a\) is a unique fixed point of \(T\).

To prove the last statement of Theorem 1 take an \(x \in X\), an integer \(s\), \(0 < s < n = n(a)\), and put

\[
a_k = d(a, T^{kn+s} x), \quad k = 0, 1, \ldots.
\]

Suppose that for some \(k\), \(a_k > a_{k-1}\). Then, using 4°, 1° and 3° (cf. the Lemma), we have

\[
\begin{align*}
\alpha(a_k, a_k, a_k-1, a_k-1, a_k-1, a_k) &< \alpha(a_k, a_k, a_k, a_k, 2a_k) < \alpha(a_k) < a_k.
\end{align*}
\]

This contradiction proves that \(a_k < a_{k-1}\), \(k = 1, 2, \ldots\). Hence, using 4° and 1°, we have

\[
a_k = d(T^n a, T^n T^{(k-1)n+s} x)
\]

\[
\leq \alpha(a_{k-1}, a_k, a_{k-1}, a_{k-1}, 2a_{k-1}) < \alpha(a_k) < a_k.
\]

This completes the proof.

Remark 2. Note that we have not assumed the continuity of \(T\).

2. As a simple consequence of Theorem 1 we obtain the following

**Theorem 2.** Let \((X, d)\) be a complete metric space, \(T: X \to X\), and \(\gamma: [0, \infty) \to [0, \infty)\). If \(\gamma\) is nondecreasing, \(\lim_{t \to \infty} (t - \gamma(t)) = 0\), \(\gamma(t) > 0\) for \(t > 0\), and for each \(x \in X\) there is a positive integer \(n = n(x)\) such that for all \(y \in X\),

\[
d(T^n x, T^n y) < \gamma(d(x, y)),
\]

then \(T\) has a unique fixed point \(a \in X\). Moreover, for each \(x \in X\), \(\lim_{k \to \infty} T^k x = a\).

Remark 3. Taking in Theorem 2 \(\gamma(t) = ct\) with \(0 < c < 1\), we obtain V. M. Sehgal’s fixed point theorem in which the assumption of the continuity of \(T\) is removed (cf. [4]).

For \(a(t_1, \ldots, t_5) = a t_1 + b t_2 + c t_3 + b t_4 + a t_5\), Theorem 1 yields the following

**Theorem 3.** Let \((X, d)\) be a complete metric space and let \(T: X \to X\) satisfies the following condition: for each \(x \in X\) there is a positive integer \(n = n(x)\) such that for all \(y \in X\),

\[
d(T^n x, T^n y) \leq d(x, T^n x) + d(y, T^n y) + b[d(x, T^n y) + d(T^n x, y)] + c d(x, y)
\]

where \(a, b, c\) are nonnegative and \(3a + 3b + c < 1\) then \(T\) has a unique fixed point \(p \in X\). Moreover, for every \(x \in X\), \(\lim_{k \to \infty} T^k x = p\).
Recently K. Iseki [2], generalizing the results of V. M. Sehgal [4] and L. Khazanchi [3], has obtained an analogous result but there $T$ is assumed to be continuous and $4a + 4b + c < 1$.

**Remark 4.** L. F. Guseman [1] noted that in [4] the continuity of $T$ is superfluous. He also gave an interesting reformulation of Sehgal's result. In a similar way we can formulate our Theorems 1–3.

**Example.** Let $X = [0, \infty)$, $d(x, y) = |x - y|$, $Tx = x/(1 + x)$, $\gamma(t) = t/(1 + t)$ for $x, y, t \in [0, \infty)$. We have

$$\lim_{n \to \infty} \gamma^n(t) = \lim_{n \to \infty} \frac{t}{1 + nt} = 0 \quad \text{for} \ t \geq 0, \ \lim_{t \to \infty} (t - \gamma(t)) = \infty$$

and

$$d(Tx, Ty) = \frac{|x - y|}{(1 + x)(1 + y)} \leq \frac{|x - y|}{1 + |x - y|} = \gamma(d(x, y)).$$

Thus all the assumptions of Theorem 2 are fulfilled, but that is not the case for Theorem 3. To see this suppose that there are nonnegative $a, b, c$ satisfying conditions of Theorem 3. Then for $x = 0$ we obtain

$$\frac{y}{1 + ny} \leq a \left(y - \frac{y}{1 + ny}\right) + b \left(\frac{y}{1 + ny} + y\right) + cy, \quad n = n(0), \ y > 0.$$ 

Hence $(a + b + c)/(1 - a + b) \geq 1/(1 + ny)$ for $y > 0$ and, consequently, $2b + c \geq 1$. This contradiction proves that Theorem 1 is stronger than the results of [1]—[4].

**Remark 5.** Suppose that $T: X \to X$ and there is a point $p \in X$ such that \(\{d(T^n x, p)\}\) tends to 0 uniformly in $X$. Fix an $a > 0$. Then for every $x \in X - \{p\}$ there exists a positive integer $n = n(x)$ such that for all $y \in X$, $d(T^n x, T^n y) \leq ad(x, T^n x)$. Using this remark one can easily construct an example of a mapping $T$ satisfying all the conditions of Theorem 3 and such that for every $n$, $T^n$ is discontinuous.

**References**


