SPIN MANIFOLDS ARE DECOMPOSABLE

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Abstract. It is shown that in the unoriented cobordism ring every manifold with Spin structure is decomposable.

1. Introduction. From the work of Thom [3], the unoriented cobordism ring \( \mathfrak{n}_* \) is a polynomial ring over \( \mathbb{Z}_2 \) on generators \( x_i \) of dimension \( i \), with \( i \neq 2^k - 1 \), and the manifold \( M^i \) is indecomposable if and only if the characteristic number \( S_i[M^i] \) is nonzero.

Dold [2] exhibited odd dimensional manifolds which are suitable generators, and these generators are in fact orientable manifolds. No even dimensional oriented manifold can be indecomposable, since

\[
S_{2n}[M^{2n}] = Sq^{1}S_{2n-1}[M^{2n}] = w_1S_{2n-1}[M^{2n}].
\]

An examination of the known examples for generators quickly reveals that none admit Spin structures. This is not a coincidence, for one has

Proposition. Every Spin manifold is decomposable.

2. Proof. Since a Spin manifold is oriented, even dimensional Spin manifolds are decomposable. Further, if \( M \) is a Spin manifold of dimension \( 4n + 1 \),

\[
S_{4n+1}[M] = Sq^2S_{4n-1}[M] = v_2S_{4n-1}[M] = 0.
\]

Now, let \( M \) be a Spin manifold of dimension \( 4n + 3 \).

Claim. If \( j \leq n \), and \( x \in H^{n-j}(M; \mathbb{Z}_2) \), then \( x^4S_{4j+3}[M] = 0 \). This is clear for \( j = 0 \), since \( S_3 \) is zero, and inductively it may be assumed if \( j' < j \).

Now \( Sq^1S_{4j+3} = S_{4j+4} = S_{2j+2} = Sq^1(S_{2j+2}S_{2j+1}) \), and by Proposition 6.1 of [1], \( ker Sq^1 \) is im \( Sq^1 \) in \( H^*(B \text{Spin}; \mathbb{Z}_2) \) except in dimensions divisible by four. Thus \( S_{4j+3} = S_{2j+2}S_{2j+1} + Sq^1 \sigma \) for some \( \sigma \), and

\[
x^4S_{4j+3}[M] = x^4S_{2j+2}S_{2j+1}[M] + Sq^1(x^4\sigma)[M] = x^4S_{2j+2}S_{2j+1}[M].
\]

If \( j \) is odd, \( 2j + 1 = 4j' + 3 \), \( j' < j \) and

\[
x^4S_{2j+2}S_{2j+1}[M] = (xS_{(j+1)/2})^4S_{4j'+3}[M] = 0.
\]

If \( j \) is even,
\[ x^4 S_{2j+2} S_{2j+1} [M] = x^4 S_{2j+2} S^2 S_{2j-1} [M] \]
\[ = x^4 S^2 (S_{2j+2}) S_{2j-1} [M] \]
\[ = x^4 S_{2j+4} S_{2j-1} [M] \]
\[ = (x S_{(j+1)/2})^4 S_{2j-1} [M], \]

which is zero, since \(2j - 1 = 4j' + 3, j' < j\).

REFERENCES


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