BINARY OPERATIONS IN THE SET OF SOLUTIONS OF A PARTIAL DIFFERENCE EQUATION

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Abstract. Let $\mathcal{O}$ be a partial difference operator with constant coefficients in $n$ independent (discrete) variables, and let $\mathcal{S}_\mathcal{O} = \{ f : \mathbb{Z}^n \to \mathbb{C}; \mathcal{O} f = 0 \}$. We introduce a certain class of binary operations $S_\mathcal{O} \times S_\mathcal{O} \to S_\mathcal{O}$ generalizing a binary operation introduced by Duffin and Rohrer.

1. Introduction. Let $\mathbb{Z}^n$ be the $n$-dimensional lattice and consider a partial difference operator on $\mathbb{Z}^n$

$$\mathcal{O} f(m) = \sum_{|k| \leq N} C_k f(m + k),$$

where $m, k \in \mathbb{Z}^n$, $|k| = \sum_{i=1}^{n} |k_i|$, $k = (k_1, \ldots, k_n)$ and $N$ is an integer. In this note we shall characterize all products $*$ of the form

$$(f * g)(m) = \sum_{r \in \mathbb{Z}^n, k \in \mathbb{Z}^n} d_{kr}^m f(r) g(k)$$

(only a finite number of terms on the right-hand side being nonzero) with the property that if $\mathcal{O} f = 0$ and $\mathcal{O} g = 0$ then $\mathcal{O}(f * g) = 0$. The product of Duffin and Rohrer [1] falls in this category. The basic idea is to associate with every discrete function $f : \mathbb{Z}^n \to \mathbb{C}$ a linear functional $T_f$ on the algebra $\mathcal{S}_\mathcal{O}$ generated by the indeterminates $(z_1, z_1^{-1}, \ldots, z_n, z_n^{-1})$, given by

$$(1.2) \quad T_f(z_1^{k_1}, \ldots, z_n^{k_n}) = f(k_1, \ldots, k_n)$$

for every $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ and extended by linearity. Conversely, (1.2) associates a discrete function $f : \mathbb{Z}^n \to \mathbb{C}$ to every such linear functional.

2. Binary operations on the set of solutions of $\mathcal{O} u = 0$.

Definition 2.1. Any operation $(f, g) \to f * g$ which maps pairs of functions on $\mathbb{Z}^n$ to another function on $\mathbb{Z}^n$ and is of the form (1.1) will be termed a Duffin product.

Lemma 2.2. Any Duffin product induces a linear mapping $\mathcal{S} : \mathcal{S}_\mathcal{O} \to \mathcal{S}_{2n}$ such that if $z = (z_1, \ldots, z_n), t = (t_1, \ldots, t_n)$.

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242
BINARY OPERATIONS IN THE SET OF SOLUTIONS OF A PDE 243

(2.1) \[ T_f g (u(z)) = T_f T_g (S u(z, t)) \]
where \( T_f T_g \) is the linear functional on \( \mathcal{A}_{2n} \) defined by

(2.2) \[ T_f T_g (z^k t^r) = T_f (z^k) T_g (t^r) \]

and extended by linearity.

**Proof.** By (1.1)

\[ T_f g (z^m) = (f \ast g)(m) = \sum d_{kr} T_f (z^k) T_g (t^r) = T_f T_g (\sum d_{kr} z^k t^r). \]

Define \( \mathcal{F}(z^m) = \sum d_{kr} z^k t^r \) and extend by linearity. Obviously (2.1) defines a Duffin product for each such mapping.

**Lemma 2.3.** Let \( \mathcal{G} \) be a partial difference operator with constant coefficients
\[ \mathcal{G} f (m) = \sum C_k f (m + k), \]
and let \( P(z) \in \mathcal{B}_n \) be its symbol, \( P(z) = \sum C_k z^k \).

Then \( \mathcal{G} f \equiv 0 \) iff \( T_f \) annihilates the principal ideal \( P(z) \mathcal{B}_n = \{ P(z) u(z); u(z) \in \mathcal{B}_n \} \).

**Proof.** The statement is self-evident from the identity

\[ T_f (P(z) z^m) = T_f (\sum C_k z^{m+k}) = \sum C_k f (m + k). \]

Now we are in a position to prove our central result.

**Theorem.** A Duffin product induced by the mapping \( \mathcal{G} : \mathcal{B}_n \to \mathcal{B}_{2n} \), given in
Lemma 2.2, maps pairs of solutions of \( \mathcal{G} u = 0 \) into another solution if \( \mathcal{G}(P(z) \mathcal{B}_n) \) is contained in the ideal generated by \( \{ P(z), P(t) \} \), i.e., if for every \( u(z) \in \mathcal{B}_m \) we can find \( a(z, t), b(z, t) \in \mathcal{B}_{2n} \) such that

\[ \mathcal{G} (P(z) u(z)) = a(z, t) P(z) + b(z, t) P(t). \]

**Proof.** \( \mathcal{G} (f \ast g) \equiv 0 \) if \( T_f g (P(z) \mathcal{B}_n) = 0 \). Now

\[ T_f g (P(z) u(z)) = T_f T_g (\mathcal{G} P(z) u(z)) = T_f T_g (a(z, t) P(z) + b(z, t) P(t)) = 0. \]

3. **Applications.** The theorem makes very easy the verification that a given Duffin product preserves the property of being a solution of a given partial difference equation with constant coefficients. This will be illustrated by the following two examples.


(3.1) \[ f(m, n) + if(m + 1, n) - f(m + 1, n + 1) - if(m, n + 1) = 0. \]

They denoted them by \( f \ast g, f \ast' g \) and \( f \ast'' g \). An easy calculation, which is not reproduced here in order to save space, shows that the corresponding mappings \( \mathcal{F}, \mathcal{F}', \mathcal{F}'' : \mathcal{B}_2 \to \mathcal{B}_4 \) are (make the notational transformation \( z = (z_1, z_2) = (z, w), t = (t_1, t_2) = (t, s) \))
\[ \mathcal{T}: u(z, w) \to (1 + i)(1 + z) \frac{u(z, w) - u(t, w)}{z - t} \\
+ i(1 + s)(1 + w) \frac{u(t, w) - u(t, s)}{w - s}, \]

\[ \mathcal{T}': u(z, w) \to (1 + z)(1 - t) \frac{u(z, w) - u(t, w)}{z - t} \\
+ i(1 - s)(1 + w) \frac{u(t, w) - u(t, s)}{w - s}, \]

\[ \mathcal{T}'': u(z, w) \to (1 - z)(1 - t) \frac{u(z, w) - u(t, w)}{z - t} \\
+ i(1 - s)(1 - w) \frac{u(t, w) - u(t, s)}{w - s}. \]

From these formulas we deduce easily that the corresponding convolution products indeed preserve discrete-analyticity (i.e., the property of being a solution of (3.1)). They can also be used to advantage in giving short proofs of the commutativity and associativity of these products.

(b) For a general partial difference equation with constant coefficients \( \mathcal{P}u = 0 \), in \( \mathbb{Z}^2 \), Duffin and Rohrer [1] introduced a 'product' which can be shown, by a straightforward but a little lengthy calculation, to be induced by

\[ \mathcal{T}(u(z, w)) = ts \left\{ \frac{u(t, s) - u(t, w)}{s - w} \left[ \frac{P(z, w) - P(t, w)}{z - t} \right] \\
- \frac{u(z, w) - u(t, w)}{z - t} \left[ \frac{P(t, s) - P(t, w)}{s - w} \right] \right\} \]

\[ = \frac{ts}{(s - w)(z - t)} [u(t, s)] [P(z, w) - P(t, w)] \\
- u(t, w) [P(z, w) - P(t, s)] \\
- u(z, w) [P(t, s) - P(t, w)], \]

where \( P(z, w) \) is the symbol of \( \mathcal{P} \). \( \mathcal{T} \) is seen to satisfy the hypothesis of the Theorem, thus furnishing a short proof to the fact that if \( \mathcal{P}f = 0 \) and \( \mathcal{P}g = 0 \), then \( \mathcal{P}(f \ast g) = 0 \) (see Duffin and Rohrer [1, pp. 691–693] for the original proof).

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