

BINARY OPERATIONS IN THE SET OF SOLUTIONS OF A PARTIAL DIFFERENCE EQUATION

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ABSTRACT. Let \mathcal{P} be a partial difference operator with constant coefficients in n independent (discrete) variables, and let $\mathcal{S}_{\mathcal{P}} = \{f: Z^n \rightarrow \mathbb{C}; \mathcal{P}f = 0\}$. We introduce a certain class of binary operations $\mathcal{S}_{\mathcal{P}} \times \mathcal{S}_{\mathcal{P}} \rightarrow \mathcal{S}_{\mathcal{P}}$ generalizing a binary operation introduced by Duffin and Rohrer.

1. Introduction. Let Z^n be the n -dimensional lattice and consider a partial difference operator on Z^n

$$\mathcal{P}f(m) = \sum_{|k| \leq N} C_k f(m+k),$$

where $m, k \in Z^n$, $|k| = \sum_{i=1}^n |k_i|$, $k = (k_1, \dots, k_n)$ and N is an integer. In this note we shall characterize all products $*$ of the form

$$(1.1) \quad (f * g)(m) = \sum_{r \in Z^n; k \in Z^n} d_{kr}^m f(r)g(k)$$

(only a finite number of terms on the right-hand side being nonzero) with the property that if $\mathcal{P}f \equiv 0$ and $\mathcal{P}g \equiv 0$ then $\mathcal{P}(f * g) \equiv 0$. The product of Duffin and Rohrer [1] falls in this category. The basic idea is to associate with every discrete function $f: Z^n \rightarrow \mathbb{C}$ a linear functional T_f on the algebra \mathcal{Q}_n generated by the indeterminates $\{z_1, z_1^{-1}, \dots, z_n, z_n^{-1}\}$, given by

$$(1.2) \quad T_f(z_1^{k_1}, \dots, z_n^{k_n}) = f(k_1, \dots, k_n)$$

for every $(k_1, \dots, k_n) \in Z^n$ and extended by linearity. Conversely, (1.2) associates a discrete function $f: Z^n \rightarrow \mathbb{C}$ to every such linear functional.

2. Binary operations on the set of solutions of $\mathcal{P}u \equiv 0$.

DEFINITION 2.1. Any operation $(f, g) \rightarrow f * g$ which maps pairs of functions on Z^n to another function on Z^n and is of the form (1.1) will be termed a *Duffin product*.

LEMMA 2.2. Any Duffin product induces a linear mapping $\mathcal{F}: \mathcal{Q}_n \rightarrow \mathcal{Q}_{2n}$ such that if $z = (z_1, \dots, z_n)$, $t = (t_1, \dots, t_n)$,

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$$(2.1) \quad T_{f * g}(u(z)) = T_f T_g(\mathfrak{F}u(z, t))$$

where $T_f T_g$ is the linear functional on \mathcal{Q}_{2n} defined by

$$(2.2) \quad T_f T_g(z^k t^r) = T_f(z^k) T_g(t^r)$$

and extended by linearity.

PROOF. By (1.1)

$$T_{f * g}(z^m) = (f * g)(m) = \sum d_{kr}^m T_f(z^k) T_g(t^r) = T_f T_g(\sum d_{kr}^m z^k t^r).$$

Define $\mathfrak{F}(z^m) = \sum d_{kr}^m z^k t^r$ and extend by linearity. Obviously (2.1) defines a Duffin product for each such mapping.

LEMMA 2.3. Let \mathfrak{P} be a partial difference operator with constant coefficients $\mathfrak{P}f(m) = \sum C_k f(m + k)$, and let $P(z) \in \mathcal{Q}_n$ be its symbol, $P(z) = \sum C_k z^k$. Then $\mathfrak{P}f \equiv 0$ iff T_f annihilates the principal ideal $P(z)\mathcal{Q}_n = \{P(z)u(z); u(z) \in \mathcal{Q}_n\}$.

PROOF. The statement is self-evident from the identity

$$T_f(P(z)z^m) = T_f(\sum C_k z^{m+k}) = \sum C_k f(m + k).$$

Now we are in a position to prove our central result.

THEOREM. A Duffin product induced by the mapping $\mathfrak{F}: \mathcal{Q}_n \rightarrow \mathcal{Q}_{2n}$, given in Lemma 2.2, maps pairs of solutions of $\mathfrak{P}u \equiv 0$ into another solution if $\mathfrak{F}(P(z)\mathcal{Q}_n)$ is contained in the ideal generated by $\{P(z), P(t)\}$, i.e., if for every $u(z) \in \mathcal{Q}_n$ we can find $a(z, t), b(z, t) \in \mathcal{Q}_{2n}$ such that

$$\mathfrak{F}(P(z)u(z)) = a(z, t)P(z) + b(z, t)P(t).$$

PROOF. $\mathfrak{P}(f * g) \equiv 0$ if $T_{f * g}(P(z)\mathcal{Q}_n) = 0$. Now

$$T_{f * g}(P(z)u(z)) = T_f T_g(\mathfrak{F}P(z)u(z)) = T_f T_g(a(z, t)P(z) + b(z, t)P(t)) = 0.$$

3. Applications. The theorem makes very easy the verification that a given Duffin product preserves the property of being a solution of a given partial difference equation with constant coefficients. This will be illustrated by the following two examples.

(a) Duffin and Duris [2] introduced three kinds of 'convolution products' for solutions of the discrete Cauchy-Riemann equation.

$$(3.1) \quad f(m, n) + if(m + 1, n) - f(m + 1, n + 1) - if(m, n + 1) \equiv 0.$$

They denoted them by $f * g, f *' g$ and $f *'' g$. An easy calculation, which is not reproduced here in order to save space, shows that the corresponding mappings $\mathfrak{F}, \mathfrak{F}', \mathfrak{F}'' : \mathcal{Q}_2 \rightarrow \mathcal{Q}_4$ are (make the notational transformation $z = (z_1, z_2) = (z, w), t = (t_1, t_2) = (t, s)$)

$$\begin{aligned} \mathfrak{F}: u(z, w) &\rightarrow (1+t)(1+z) \frac{u(z, w) - u(t, w)}{z-t} \\ &\quad + i(1+s)(1+w) \frac{u(t, w) - u(t, s)}{w-s}, \\ \mathfrak{F}': u(z, w) &\rightarrow (1+z)(1-t) \frac{u(z, w) - u(t, w)}{z-t} \\ &\quad + i(1-s)(1+w) \frac{u(t, w) - u(t, s)}{w-s}, \\ \mathfrak{F}'': u(z, w) &\rightarrow (1-z)(1-t) \frac{u(z, w) - u(t, w)}{z-t} \\ &\quad + i(1-s)(1-w) \frac{u(t, w) - u(t, s)}{w-s}. \end{aligned}$$

From these formulas we deduce easily that the corresponding convolution products indeed preserve discrete-analyticity (i.e., the property of being a solution of (3.1)). They can also be used to advantage in giving short proofs of the commutativity and associativity of these products.

(b) For a general partial difference equation with constant coefficients $\mathcal{P}u \equiv 0$, in Z^2 , Duffin and Rohrer [1] introduced a 'product' which can be shown, by a straightforward but a little lengthy calculation, to be induced by

$$\begin{aligned} \mathfrak{F}(u(z, w)) &= ts \left\{ \frac{u(t, s) - u(t, w)}{s-w} \left[\frac{P(z, w) - P(t, w)}{z-t} \right] \right. \\ &\quad \left. - \frac{u(z, w) - u(t, w)}{z-t} \left[\frac{P(t, s) - P(t, w)}{s-w} \right] \right\} \\ &= \frac{ts}{(s-w)(z-t)} [u(t, s)[P(z, w) - P(t, w)] \\ &\quad - u(t, w)[P(z, w) - P(t, s)] \\ &\quad - u(z, w)[P(t, s) - P(t, w)], \end{aligned}$$

where $P(z, w)$ is the symbol of \mathcal{P} . \mathfrak{F} is seen to satisfy the hypothesis of the Theorem, thus furnishing a short proof to the fact that if $\mathcal{P}f \equiv 0$ and $\mathcal{P}g \equiv 0$, then $\mathcal{P}(f * g) \equiv 0$ (see Duffin and Rohrer [1, pp. 691–693] for the original proof).

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