UNIQUE HAHN-BANACH EXTENSIONS AND SIMULTANEOUS EXTENSIONS

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ABSTRACT. In this note we mention certain connections existing between unique Hahn-Banach extensions and simultaneous extensions. We also describe an application of a continuous selection theorem to simultaneous extensions.

1. Effects of unique Hahn-Banach extensions. Let \((m)\) be the Banach space of bounded sequences of real (or complex) numbers with the supremum norm and \((c_0)\) the subspace of \((m)\) consisting of sequences converging to zero. Then the following holds, which is ascribed to H. P. Rosenthal (cf. Banilower [1, Proposition 1.2]): If \(T\) is a linear operator from \((m)\) into a normed linear space \(F\) with \(\|T\| = 1\) such that \(T\) is an isometry on \((c_0)\), then \(T\) is an isometry on \((m)\). If, moreover, \(F = (m)\) and \(T\) is the identity operator on \((c_0)\), then \(T\) is the identity operator on \((m)\). Our first objective is to describe principles underlying this result.

Let \(E\) be a normed linear space with dual \(E^*\), \(L\) a subspace of \(E\), \(T\) a linear operator from \(E\) into a normed linear space \(F\) with \(\|T\| = 1\) such that \(T\) is an isometry on \(L\), and \(M = T(L)\), which is considered as a subspace of \(F\). These notations are preserved through this section. Let \(S^*(E, L)\) be the set of \(x^* \in E^*\) with \(\|x^*\| = 1\) such that \(x^*|L = y^*|L\) implies \(x^* = y^*\) if \(\|y^*\| \leq 1\). If \(x^* \in S^*(E, L)\), then \(\|x^*|L\| = 1\) and \(x^*\) is the unique Hahn-Banach extension of \(x^*|L\) to \(E\). We denote by \(E^*(L)\) the subspace of \(E^*\) generated by the \(\sigma(E^*, E)-\)closed convex hull of \(S^*(E, L)\) and by \(E^*(L)\) the orthogonal complement of \(E^*(L)\) in the space \(E\). By the Krein-Šmulian theorem (cf. Dunford [3, V.5.8]), \(E^*(L)\) is \(\sigma(E^*, E)-\)closed. For a given subset \(D\) of \(E\) we say that an element \(x \in E\) is orthogonal to \(D\) if \(\|x + cx'\| \geq \|x\|\) for any \(x' \in D\) and any scalar \(c\). Consider the case in which \(D\) is a closed subspace. Since \(D^⊥\), the orthocomplement of \(D\) in \(E^*\), is regarded as the dual of the quotient space \(E/D\), we have

\[
\|x\| \geq \inf\{\|x + x'\| : x' \in D\} = \sup\{\|\langle x, x^* \rangle\| : x^* \in D^⊥ \text{ and } \|x^*\| = 1\}.
\]
The equality sign prevails if and only if \( x \) is orthogonal to \( D \). As for the isometric property of \( T \) we have the following

**Theorem 1.** \(|T_x| = |x|\) for \( x \in E \), if either \( x \in L \) or \( x \) is orthogonal to the subspace \( E^*(L)^\perp \).

**Proof.** Take any \( x^* \in S^*(E,L) \) and set \( v^* = x^* \circ (T|L)^{-1} \). Then \( v^* \) is a linear functional on the subspace \( M = T(L) \) and \(|v^*| = |x^*| |L| = |x^*| = 1\), for \( T \) is an isometry on \( L \). Let \( y^* \) be any Hahn-Banach extension of \( v^* \) to \( F \). \( y^* \circ T \) is then a Hahn-Banach extension of \( x^* |L \) and therefore \( x^* = y^* \circ T \). For any \( x \in E \) we have \(|T_x| \geq |\langle T_x,y^* \rangle| = |\langle x,y^* \circ T \rangle| = |\langle x,x^* \rangle| \) and so \(|T_x| \geq \sup \{ |\langle x,x^* \rangle| : x^* \in S^*(E,L) \}\). If \( x \) is orthogonal to \( E^*(L)^\perp \), then the above remark with \( D = E^*(L)^\perp \) shows that

\[
|v| = \sup \{ |\langle x,x^* \rangle| : x^* \in S^*(E,L), |x^*| = 1 \}
\]

isometries. If \( x \in L \), then \(|F_x| = |x|\), for \( F \) is an isometry on \( L \).

**Corollary 2.** If \( E^*(L) = E^* \), then \( T \) is an isometry on \( E \).

After Phelps [7] we say that a subspace \( L \) of \( E \) has property \( U \) in \( E \) if every bounded linear functional on \( L \) has a unique Hahn-Banach extension to \( E \), i.e., \( S^*(E,L) \) coincides with the set of \( x^* \in E^* \) for which \(|x^*| = |x^*| |L| = 1\).

**Corollary 3.** If \( L \) has property \( U \) in \( E \) and if \( E \subseteq L^{**} \), then \( T \) is an isometry on \( E \).

We next consider another aspect of Rosenthal's result. For a subset \( \mathcal{L} \) of \( L^* \) let \( H(E,\mathcal{L}) \) be the set of \( x \in E \) such that all Hahn-Banach extensions from \( L \) to \( E \) of any element in \( \mathcal{L} \) coincide at \( x \). \( H(E,\mathcal{L}) \) is the largest subspace of \( E \) containing \( L \) to which every element in \( \mathcal{L} \) has a unique Hahn-Banach extension.

**Theorem 4.** Let \( \mathcal{L} \subseteq L^* \) be such that the set of \( y^* \in F^* \) for which \((y^* \circ T)|L \in \mathcal{L} \) and \(|y^*| = |y^*| |M| \) separates points of \( F \). If \( T_1 \) is a linear operator from \( E \) into \( F \) such that \(|T_1| = 1 \) and \( T_1|L = T|L \), then \( T_1(x) = T(x) \) for any \( x \in H(E,\mathcal{L}) \).

**Proof.** Let \( x \in H(E,\mathcal{L}) \) and suppose, on the contrary, that \( T_1(x) \neq T(x) \). By the hypothesis on \( F \) there exists a \( y^* \in F^* \) such that \((y^* \circ T)|L \in \mathcal{L} \) and \(|y^*| = |y^*| |M| \) and \( \langle T_1(x),y^* \rangle \neq \langle T(x),y^* \rangle \). Since \( y^* \circ T \) and \( y^* \circ T_1 \) are Hahn-Banach extensions to \( E \) of the functional \((y^* \circ T)|L \in \mathcal{L} \), they coincide at the point \( x \), which contradicts the choice of \( y^* \).

**Corollary 5.** If \( E = H(E,L^*) \) and the set of \( y^* \in F^* \) with \(|y^*| = |y^*| |M| \) separates points of \( F \), then \( T \) is uniquely determined by its values on \( L \).
The condition on $F$ given in this corollary is satisfied if $F \subseteq M^{**}$, because $M^*$ separates points of $M^{**}$ and the norm of each $\nu^* \in M^*$ is the same as the norm of $\nu^*$ as a linear functional on $M^{**}$ (and so as a linear functional on $F$).

**Corollary 6.** Suppose that (i) $L$ has property $U$ in $E$, (ii) $E^*(L) = E^*$, (iii) the set of $y^* \in F^*$ with $\|y^*\| = \|\nu^*|_M\|$ separates points of $F$. Then $T$ is an isometry and is determined by its values on $L$.

**Example 1.** Let $\Sigma$ be a compact Hausdorff space and $\Sigma_0$ a closed subset of $\Sigma$. If $C(\Sigma)$ denotes the Banach space of continuous real (or complex) functions on $\Sigma$ with the supremum norm, then the subspace $C(\Sigma|\Sigma_0) = \{f \in C(\Sigma) : f|\Sigma_0 = 0\}$ has property $U$ in $C(\Sigma)$ (cf. Phelps [7]). If $\Sigma|\Sigma_0$ is dense in $\Sigma$, then it is easy to see that $C(\Sigma)$ is contained in the second dual of $C(\Sigma|\Sigma_0)$. If $\beta N$ denotes the Čech compactification of the set $N$ of positive integers, then $(m)$ is isometrically isomorphic with $C(\beta N)$ and $(c_0)$ with $C(\beta N|\beta N \setminus N)$. Hence $(c_0)$ has property $U$ in $(m)$ and moreover $(m) = (c_0)^{**}$. So Rosenthal's result cited above follows from Corollaries 2 and 6.

Concerning property $U$, we know that every two-sided ideal $I$ of a $C^*$-algebra $A$ has this property in $A$, of which the pair $((c_0),(m))$ is a special case. A much stronger result is contained in Dixmier [2, Proposition 2.11.7]. It is also seen that every hereditary subalgebra of a $C^*$-algebra $A$ has property $U$ in $A$.

**Example 2.** We need neither the property $U$ for $L$ nor the fact like $E \subseteq L^{**}$ in order to assert that $T$ is an isometry. This is illustrated by the well-known example in Korovkin's theory of approximation. Let $E = C([0,1])$ be the space of all continuous real (or complex) functions on the closed interval $[0,1]$ and $L$ the subspace of $E$ spanned by three functions; 1, $t$ and $t^2$. Although $L$ does not have property $U$ in $E$, every evaluation functional $\varepsilon_a : f \to f(a)$ with $a \in [0,1]$ on the space $L$ has a unique Hahn-Banach extension, namely $\varepsilon_{a,t}$ to $E$. Since the set $\delta = \{\varepsilon_a : a \in [0,1]\}$ coincides with the set of extreme points of the unit ball of $E^*$ and is contained in $\delta^*(E,L)$, we see that $E^*(L) = E^*$. So Corollary 2 applies to this case. On the other hand, we have $E = H(E,\delta|L)$. If the set of $y^* \in F^*$ for which $y^* \circ T = \varepsilon_a$ on $L$ for some $a \in [0,1]$ separates points of $F$, then $T$ is uniquely determined by its values on $L$. In particular, if $T$ is a linear operator from $C([0,1])$ into itself with $\|T\| = 1$ and if $T(1) = 1, T(t) = t$ and $T(t^2) = t^2$, then $T$ is the identity operator on $C([0,1])$.

**Example 3.** Let $E = (m)$. Then the subspace $(c_0)$ has a proper closed subspace $L$ for which $E^*(L) = E^*$. Let $(a_n : n = 1, 2, \ldots)$ be a sequence such that, for each $k$, $|a_k| < \sum_{n \neq k} |a_n| < \infty$ and $L$ the set of all sequences $x = (x_1, x_2, \ldots)$ in $(c_0)$ with $\sum_{n=1}^{\infty} a_n x_n = 0$. It is clear that $L$ is a closed subspace of $(c_0)$. It is also easy to see that each evaluation functional $\varepsilon_n : x \to x_n$ has a unique Hahn-Banach extension to $(c_0)$ and consequently to $E$. It follows that $E^*(L) = E^*$ and Corollary 2 applies to this $L$. $L$ does not contain any nontrivial ideal of $(c_0)$ (or $(m)$) and thus does not have property
U in \((c_0)\) (or \((m)\)). This last fact has been observed by S. Takahasi in a more general situation. It is seen also that \(L^{**}\) is strictly smaller than \((m)\).

Here we include the following very slight modification of Kurtz’s theorem [6, Theorem 3].

**Proposition 7.** Let \(S\) be a subset of the unit sphere \(\{y^* \in F^* : \|y^*\| = 1\}\) in \(F^*\) such that \(y^* \circ T \in S^*(E, L)\) for any \(y^* \in S\). Let \(\{T_x\}\) be a net of linear operators from \(E\) into \(F\) with \(\|T_x\| \leq 1\) such that \(\|T_x x - Tx\| \to 0\) for all \(x \in L\). Then, for each \(x \in E\), \(\langle T_x x, y^* \rangle \to \langle Tx, y^* \rangle\) uniformly on all \(\sigma(F^*, F)\)-compact subsets of \(S\). In particular, if \(S\) contains the \(\sigma(F^*, F)\)-closure of the extreme points of the unit sphere in \(F^*\), then \(\|T_x x - Tx\| \to 0\) for each \(x \in E\).

2. Simultaneous extensions. Let \(\Sigma\) be a completely regular Hausdorff space and \(C(\Sigma)\) the Banach space of all bounded continuous real (or complex) functions on \(\Sigma\) with the supremum norm. Let \(\Omega\) be a subspace of \(\Sigma\). If \(X\) and \(Y\) are subspaces of \(C(\Omega)\) and \(C(\Sigma)\), respectively, then a simultaneous extension is, by definition, a linear bounded operator \(T\) from \(X\) into \(Y\) such that \(T(f)|\Omega = f\) for all \(f \in X\). The result of Phelps cited above implies the following, which extends Banilower [1, Corollary 1.3].

**Proposition 8.** Let \(\Sigma\) be a completely regular Hausdorff space and \(\Omega\) a locally compact subspace of \(\Sigma\). Let \(C_0(\Omega)\) be the subspace of \(C(\Omega)\) consisting of elements \(f\) which vanish at infinity. If \(T\) is a linear operator from \(C(\Omega)\) into \(C(\Sigma)\) with \(\|T\| = 1\) and \(T|C_0(\Omega)\) is a simultaneous extension, then \(T\) is a simultaneous extension.

**Proof.** Let \(R: C(\Sigma) \to C(\Omega)\) be the restriction operator. Then \(R \circ T\) maps \(C(\Omega)\) into \(C(\Omega)\). Our assumption says that \(\|R \circ T\| = 1\) and \(R \circ T\) induces the identity operator on \(C_0(\Omega)\). By Corollary 5, \(R \circ T\) is the identity operator, as was to be proved.

The examples in §1 furnish other kinds of simultaneous extensions. For instance we have

**Proposition 9.** Let \(\varphi\) be a homeomorphism of \([0, 1]\) into a completely regular Hausdorff space \(\Sigma\) and \(T\) a linear operator from \(C([0, 1])\) into \(C(\Sigma)\) with \(\|T\| \leq 1\). If \(T\) induces a simultaneous extension on the space spanned by \(1, t, t^2\) in the sense that \((Tf)(\varphi(t)) = f(t)\) for \(f = 1, t, t^2\), then the same equality holds for any \(f \in C([0, 1])\).

Finally we extend [1, Proposition 1.4 and Theorem 1.5].

**Lemma 10.** Let \(\Sigma\) be a compact Hausdorff space and \(\Omega\) a completely regular Hausdorff space which is extremally disconnected in the sense that the closure of every open set in \(\Omega\) is open. If \(V\) is a linear isometric operator from \(C(\Omega)\) into \(C(\Sigma)\), then there exists a homeomorphism \(\pi\) from \(\Omega\) into \(\Sigma\) such that, for any \(f \in C(\Omega)\) and any \(x \in \Omega\),

\[
(1) \quad \|(V(1))(\pi(x))\| = 1,
\]
Proof. Since $V$ is an isometry, the transposed mapping $V^*$ of $V$ maps the closed unit ball $B^*(\Sigma)$ of the dual $C(\Sigma)^*$ onto the closed unit ball $B^*(\Omega)$ of $C(\Omega)^*$. For each $x \in \Omega$ let $\varepsilon_x$ be the evaluation functional $f \rightarrow f(x)$ on $C(\Omega)$. Since $C(\Omega)$ and $C(\beta \Omega)$ are isometrically isomorphic under the canonical mapping, we see that $\varepsilon_x$ is an extreme point of the ball $B^*(\Omega)$. So the set \((V^*)^{-1}(\varepsilon_x) \cap B^*(\Sigma) (= K(x), \text{say})\) is a support of $B^*(\Sigma)$, which is convex and weakly* compact. Let $\mu$ be an extreme point of $K(x)$, which exists by the Krein-Milman theorem. Since $K(x)$ is a support of $B^*(\Sigma)$, $\mu$ is an extreme point of $B^*(\Sigma)$, so that there exist a point $y \in \Sigma$ and a number $\alpha$, $|\alpha| = 1$, satisfying $\mu = \alpha^{-1} \varepsilon_y$ in view of [4, Lemma 7]. Thus, for each $x \in \Omega$, there exist a point $y \in \Sigma$ and a number $\alpha$ with $|\alpha| = 1$ such that
\[
(V(f))(y) = \langle V(f), \varepsilon_y \rangle = \langle f, \alpha V^*(\mu) \rangle = \langle f, \alpha \varepsilon_y \rangle = \langle f, \alpha \varepsilon_x \rangle = \alpha f(x)
\]
for all $f \in C(\Omega)$. We define, for each $x \in \Omega$, $\psi(x)$ to be the set of all $y \in \Sigma$ such that there exists a number $\alpha(y)$ with $|\alpha(y)| = 1$ satisfying $\langle V(f))(y) = \alpha(y) f(y)$ for all $f \in C(\Omega)$. It is easy to see that $\psi(x)$ is closed for every $x \in \Omega$ and the mapping $\psi: x \rightarrow \psi(x)$ is an upper semicontinuous mapping from $\Omega$ into the family of nonvoid compact subsets of $\Sigma$, i.e., $\{x \in \Omega: \psi(x) \subseteq \Sigma'\}$ is open in $\Omega$ if $\Sigma'$ is open in $\Sigma$. By use of a continuous selection theorem [5, Theorem 1.1] we can find a continuous mapping $\pi$ from $\Omega$ into $\Sigma$ such that $\pi(x) \in \psi(x)$ for any $x \in \Omega$. Since the subsets $\psi(x)$ are mutually disjoint, $\pi$ is one-to-one. We have shown that, for any $f \in C(\Omega)$ and any $x \in \Omega$, \((V(f))(\pi(x)) = \alpha(\pi(x)) f(x)\). If $f = 1$, then we have \((V(1))(\pi(x)) = \alpha(\pi(x))\) for any $x \in \Omega$. Since $|\alpha(y)| = 1$, we have proved the statements (1) and (2). Finally let $\{x_\lambda\}$ be a net in $\Omega$, $x_\lambda \in \Omega$ and suppose that $\pi(x_\lambda)$ tend to $\pi(x)$. Then (2) implies that $f(x_\lambda)$ tend to $f(x)$ for any $f \in C(\Omega)$. Since $\Omega$ is completely regular, we see that $x_\lambda \rightarrow x$ in $\Omega$. Hence $\pi$ is a homeomorphism.

Theorem 11. Let $\Sigma$ be a compact Hausdorff space and $\Omega$ an extremely disconnected, completely regular Hausdorff space. Then $C(\Sigma)$ contains a subspace isometrically isomorphic to $C(\Omega)$ if and only if there exists a subspace $\Sigma_0$ of $\Sigma$ such that $\Sigma_0$ is homeomorphic with $\Omega$ and there is a simultaneous extension $T$ from $C(\Sigma_0)$ into $C(\Sigma)$ with norm one.

Proof. We have only to prove the necessity of the theorem. Let $V$ be a linear isometric mapping from $C(\Omega)$ into $C(\Sigma)$. Then there exists a homeomorphism $\pi$ from $\Omega$ into $\Sigma$ satisfying the condition of the preceding lemma. We set $\Sigma_0 = \pi(\Omega)$. Define $Q: C(\Sigma_0) \rightarrow C(\Omega)$ by setting
\[
(Q(g))(x) = g(\pi(x))/(V(1))(\pi(x)).
\]
We see that $Q$ is a linear isometry from $C(\Sigma_0)$ onto $C(\Omega)$ and therefore that
$T = V \circ Q$ is an isometry from $C(\Sigma_0)$ into $C(\Sigma)$, which is easily seen to be a simultaneous extension from $C(\Sigma_0)$ into $C(\Sigma)$, as was to be proved.

This theorem is reduced to [1, Theorem 1.5], when $\Omega$ is the discrete space of positive integers.

REFERENCES


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