

A RATHER CLASSLESS MODEL

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ABSTRACT. Using \diamond_{ω_1} , a model \mathfrak{R} of Peano arithmetic is constructed which has a unique extension (\mathfrak{R}, χ) to a model of Δ_1^1 -PA, Peano arithmetic with Δ_1^1 -comprehension.

As in Barwise-Schlipf [2], " Δ_1^1 -PA" will refer to the theory PA (Peano arithmetic) with the axiom of induction for classes, and the axiom scheme of Δ_1^1 -comprehension which asserts the universal closure of

$$\forall x(\Phi \leftrightarrow \Psi) \rightarrow \exists X \forall x(x \in X \leftrightarrow \Phi(x))$$

for all "essentially Π_1^1 -formulas" $\Phi(x)$ and all "essentially Σ_1^1 formulas" $\Psi(x)$. An essentially Π_1^1 formula is one in the smallest class containing all first-order formulas and closed under \wedge , \vee , $\forall x_i$, $\exists x_i$, and $\forall X_i$; similarly for an essentially Σ_1^1 formula. In the paper mentioned above, Barwise and Schlipf show that the nonstandard models \mathfrak{R} of PA which can be expanded to models (\mathfrak{R}, χ) of Δ_1^1 -PA are exactly the recursively saturated ones; in particular, if $DF_{\mathfrak{R}} = \{X \subseteq N : X \text{ is definable}^2 \text{ with parameters in } \mathfrak{R}\}$, then for recursively saturated models \mathfrak{R} of PA, $(\mathfrak{R}, DF_{\mathfrak{R}}) \models \Delta_1^1$ -PA. They also observe that if \mathfrak{R} is a countable recursively saturated model of PA, then $\cup\{\chi : (\mathfrak{R}, \chi) \models \Delta_1^1$ -PA\} has power 2^{\aleph_0} , using the version of Makkai's theorem in [1, IV.4]. Answering a question of Barwise, with the help of some much-appreciated conversations with him, we show that assuming \diamond_{ω_1} , there is a (recursively saturated) model \mathfrak{R} of PA such that $(\mathfrak{R}, DF_{\mathfrak{R}})$ is the only expansion of \mathfrak{R} to a model of Δ_1^1 -PA. This follows as the corollary of the following

THEOREM (\diamond_{ω_1}). *Suppose \mathfrak{M} is a countable recursively saturated model of Peano arithmetic. Then \mathfrak{M} has an ω_1 -like, recursively saturated, elementary end extension \mathfrak{N} with the following property:*

If $S \subseteq N$ such that for each $n \in N$, $\{x \in S : x <^{\mathfrak{N}} n\}$ is definable with parameters in \mathfrak{N} , then S is definable with parameters in \mathfrak{N} .

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¹ Henceforth, $|\mathfrak{R}| = N$, $|\mathfrak{R}_\alpha| = M_\alpha$, and so forth.

² "Definable" means "first-order definable".

COROLLARY (\diamond_{ω_1}). *There is a recursively saturated model \mathfrak{M} of PA such that if $(\mathfrak{M}, \chi) \models \Delta_1^1\text{-PA}$, then $\chi = DF_{\mathfrak{M}}$.*

PROOF OF COROLLARY. Let \mathfrak{M} be any countable recursively saturated model of PA, and let \mathfrak{N} be as in the conclusion of the theorem. Certainly $DF_{\mathfrak{M}} \subseteq \chi$, since $\mathfrak{N} \models \text{PA}$. To show $\chi \subseteq DF_{\mathfrak{M}}$, let $X \in \chi$. Then by using Gödel's β function, for example, we see by induction that there is a code for $\{x \in X: x <^{\mathfrak{M}} n\}$ for each $n \in N$ (via some uniform coding). By the theorem, $X \in DF_{\mathfrak{M}}$, as desired.

PROOF OF THEOREM. The proof uses the method of Keisler and Kunen [3] and the following results which can be found as indicated in Barwise [1]: Barwise Compactness Theorem (III5.6); Barwise (Extended) Completeness Theorem (III5.7); (Schlipf) \mathfrak{M} is recursively saturated if and only if $HYP_{\mathfrak{M}}$, the least admissible set with \mathfrak{M} as a set of urelements, has ordinal ω (IV5.3); and for recursively saturated \mathfrak{M} , $DF_{\mathfrak{M}} = \{X \subseteq N: X \in HYP_{\mathfrak{M}}\}$ (II7.2).

Let $\langle S_\alpha: \alpha < \omega_1 \rangle$ be as in the definition of \diamond_{ω_1} . We can assume that \mathfrak{M} has universe ω . We construct (as in Keisler [3]) an elementary end extension chain of recursively saturated models

$$\mathfrak{M} = \mathfrak{M}_0 \leq_e \mathfrak{M}_1 \leq_e \dots \leq_e \mathfrak{M}_\alpha \leq_e \dots \quad (\alpha < \omega_1),$$

where $M_\alpha = \alpha$ for limit ordinals α , by induction on α . Unless $\alpha = \gamma + 1$ for some limit ordinal γ , we set $\mathfrak{M}_\alpha = \bigcup_{\beta < \alpha} \mathfrak{M}_\beta$, a process which preserves elementarity, end extensions, and recursive saturation. If $\alpha = \gamma + 1$ for some limit γ , and S_γ is definable in \mathfrak{M}_γ , let L be a language containing constant symbols \bar{n} for all $n \in \mathfrak{M}_\gamma$ and \bar{m} for all $m \in \omega$, in an effective way, along with new constant symbols c and d , binary function symbols $+$ and \cdot , and a binary relation symbol \in . Consider the following theory T formulated in the fragment of $L_{\infty\omega}$ given by $HYP_{\mathfrak{M}_\gamma}$:

- KPU^+ (admissible set theory with a set of all urelements);
- The first-order elementary diagram of \mathfrak{M} (with urelement variables);
- $\{\bar{n} < c: n \in M_\gamma\}$;
- $\bigwedge_{n \in M_\gamma} \forall p (p < \bar{n} \rightarrow \bigvee_{r \in M_\gamma} (p = \bar{r}))$;
- " d is a finite ordinal";
- $\{\bar{m} \in d: m \in \omega\}$.

T is obviously Σ_1 on $HYP_{\mathfrak{M}_\gamma}$. Also, any $HYP_{\mathfrak{M}_\gamma}$ -finite subset of T contains only finitely many sentences of the form " $d \in \bar{m}$ " for $m \in \omega$, and thus has a model since every model of PA has an elementary end extension and every model has a HYP . By Barwise compactness, T has a countable model; but its well-founded part, which has ordinal ω , is also admissible by the Truncation Lemma (see Barwise [1, II8.4]). Thus its urelement structure is recursively saturated: call it \mathfrak{M}_α . Without loss of generality, $M_\alpha = \gamma + \omega$ and $\mathfrak{M}_\gamma <_e \mathfrak{M}_\alpha$.

The remaining case occurs when $\alpha = \gamma + 1$, γ a limit, and S_γ is not definable in \mathfrak{M}_γ . In that case we let L and T be as before but we want to take \mathfrak{M}_α to be a model of T omitting the following set Σ of formulas:

$$\{ \text{“}\bar{n}Ex\text{”} : n \in S_\gamma \} \cup \{ \neg \text{“}\bar{n}Ex\text{”} : n \notin S_\gamma \},$$

where by “ $\bar{n}Ex$ ” we mean “ \bar{n} is in the set coded by x .” We choose our coding so that if $A \subseteq B \subseteq M_\gamma$ and A and B have codes in \mathfrak{M}_γ , then the code for A is less than the code for B . It suffices to show that T locally omits Σ . Suppose not; say

$$T \models \forall x(\varphi(x) \rightarrow \text{“}\bar{n}Ex\text{”}) \text{ for } n \in S_\gamma, \text{ and}$$

$$T \models \forall x(\varphi(x) \rightarrow \neg \text{“}\bar{n}Ex\text{”}) \text{ for } n \notin S_\gamma,$$

where $\varphi(x)$ is in the fragment of $L_{\infty\omega}$ given by $HYP_{\mathfrak{M}_\gamma}$ and $\varphi(x)$ is consistent with T . Then

$$n \in S_\gamma \Leftrightarrow T \models \forall x(\varphi(x) \rightarrow \text{“}\bar{n}Ex\text{”})$$

and

$$n \notin S_\gamma \Leftrightarrow T \models \forall x(\varphi(x) \rightarrow \neg \text{“}\bar{n}Ex\text{”}).$$

Thus S is both Σ_1 and Π_1 on $HYP_{\mathfrak{M}_\gamma}$ by Barwise (extended) completeness. Since $S \subseteq M_\gamma \in HYP_{\mathfrak{M}_\gamma}$, $S \in HYP_{\mathfrak{M}_\gamma}$ by Δ_1 -separation. Therefore (by [1, II7.2]), $S_\gamma \in DF_{\mathfrak{M}_\gamma}$, a contradiction. So Σ is locally omitted by T , and we can take \mathfrak{M}_α to be a model of T as before ($M_\alpha = \gamma + \omega$, $\mathfrak{M}_\gamma \prec_e \mathfrak{M}_\alpha$), except that here \mathfrak{M}_α also omits Σ .

Now that the chain is constructed, we check to see that $\mathfrak{N} = \bigcup_{\alpha < \omega_1} \mathfrak{M}_\alpha$ has the desired properties. \mathfrak{N} is recursively saturated because it is the union of an elementary chain of recursively saturated models. Let $S \subseteq N$ such that for each $n \in N$, $\{x \in S : x <^{\mathfrak{N}} n\}$ is definable with parameters. $\{\gamma \in \omega_1 : (\mathfrak{M}_\gamma, S \cap \mathfrak{M}_\gamma) \prec (\mathfrak{N}, S)\}$ is easily seen to be closed and unbounded in ω_1 , as is $\{\gamma \in \omega_1 : \gamma \text{ is a limit}\}$. So if

$$C = \{\gamma \in \omega_1 : \gamma \text{ is a limit and } (\mathfrak{M}_\gamma, S \cap \mathfrak{M}_\gamma) \prec (\mathfrak{N}, S)\},$$

then C is also c.u.b. Pick $\gamma \in C$ such that $S \cap \gamma = S_\gamma$, using \diamond_{ω_1} . Then $(\mathfrak{M}_\gamma, S_\gamma) \prec (\mathfrak{N}, S)$. Pick any $n \in M_{\gamma+1} - M_\gamma$. Then $\{x \in S : x <^{\mathfrak{N}} n\}$ is definable in \mathfrak{N} , so it has a code r in \mathfrak{N} . By our choice of coding, $r <^{\mathfrak{N}}$ (the code for $\{x \in N : x <^{\mathfrak{N}} n\}$). But the latter is in $M_{\gamma+1}$, so since $\mathfrak{M}_{\gamma+1} \subseteq_e \mathfrak{N}$, $r \in M_{\gamma+1}$. By construction of $\mathfrak{M}_{\gamma+1}$, S_γ must be definable in \mathfrak{M}_γ . This same definition puts S in $DF_{\mathfrak{N}}$, since $(\mathfrak{M}_\gamma, S_\gamma) \prec (\mathfrak{N}, S)$.

REMARK. The following “effective version” of part of Theorem B of Keisler [3] can be proved essentially as the above theorem is proved:

Assume \diamond_{ω_1} . Let \mathfrak{A} be a countable tree-like model which satisfies the collection scheme, and suppose \mathfrak{A} is recursively saturated. Then \mathfrak{A} has a recursively saturated end elementary extension \mathfrak{B} such that \mathfrak{B} is an ω_1 -like model and every branch of \mathfrak{B} is definable in \mathfrak{B} .

Since (as Keisler shows) models of ZF and ZF minus infinity can be viewed as tree-like models, there is an obvious analogue of the theorem (proved above for PA) for models of these theories.

[Added in proof October 19, 1976. S. Shelah has informed the author of an absoluteness result which implies that \diamond_{ω_1} can be eliminated from the hypothesis of the result here as well as in the Keisler-Kunen result. (See his AMS Notices of October 1975 and February 1976.)]

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