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A RATHER CLASSLESS MODEL

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ABSTRACT. Using $\diamond_{\omega_1}$, a model $\mathfrak{N}$ of Peano arithmetic is constructed which has a unique extension $(\mathfrak{N}, \chi)$ to a model of Δ_1^1-PA, Peano arithmetic with Δ_1^1-comprehension.

As in Barwise-Schlipf [2], "Δ_1^1-PA" will refer to the theory PA (Peano arithmetic) with the axiom of induction for classes, and the axiom scheme of Δ_1^1-comprehension which asserts the universal closure of

$$\forall x(\Phi \leftrightarrow \Psi) \rightarrow \exists X \forall x(x \in X \leftrightarrow \Phi(x))$$

for all "essentially Π_1^1-formulas" $\Phi(x)$ and all "essentially Σ_1^1 formulas" $\Psi(x)$. An essentially Π_1^1 formula is one in the smallest class containing all first-order formulas and closed under $\wedge$, $\vee$, $\forall x_i$, $\exists x_i$, and $\forall X_i$; similarly for an essentially Σ_1^1 formula. In the paper mentioned above, Barwise and Schlipf show that the nonstandard models $\mathfrak{N}$ of PA which can be expanded to models $(\mathfrak{N}, \chi)$ of Δ_1^1-PA are exactly the recursively saturated ones; in particular, if $DF_{\mathfrak{N}} = \{X \subseteq N:$[1] X is definable[2] with parameters in $\mathfrak{N}\}$, then for recursively saturated models $\mathfrak{N}$ of PA, $(\mathfrak{N}, DF_{\mathfrak{N}}) \models \Delta_1^1$-PA. They also observe that if $\mathfrak{N}$ is a countable recursively saturated model of PA, then $\bigcup\{\chi: (\mathfrak{N}, \chi) \models \Delta_1^1\text{-PA}\}$ has power $2^{\aleph_0}$, using the version of Makkai's theorem in [1, IV.4]. Answering a question of Barwise, with the help of some much-appreciated conversations with him, we show that assuming $\diamond_{\omega_1}$, there is a (recursively saturated) model $\mathfrak{N}$ of PA such that $(\mathfrak{N}, DF_{\mathfrak{N}})$ is the only expansion of $\mathfrak{N}$ to a model of Δ_1^1-PA. This follows as the corollary of the following

THEOREM ($\diamond_{\omega_1}$). *Suppose $\mathfrak{M}$ is a countable recursively saturated model of Peano arithmetic. Then $\mathfrak{M}$ has an ω_1-like, recursively saturated, elementary end extension $\mathfrak{N}$ with the following property*:

> *If $S \subseteq N$ such that for each $n \in N$, $\{x \in S: x <^{\mathfrak{N}} n\}$ is definable with parameters in $\mathfrak{N}$, then S is definable with parameters in $\mathfrak{N}$.*

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[1] Henceforth, $|\mathfrak{N}| = N$, $|\mathfrak{M}_\alpha| = M_\alpha$, and so forth.

[2] "Definable" means "first-order definable".

COROLLARY ($\diamond_{\omega_1}$). *There is a recursively saturated model* $\mathfrak{N}$ *of PA such that if* $(\mathfrak{N}, \chi) \models \Delta^1_1$-*PA, then* $\chi = DF_{\mathfrak{N}}$.

PROOF OF COROLLARY. Let $\mathfrak{M}$ be any countable recursively saturated model of PA, and let $\mathfrak{N}$ be as in the conclusion of the theorem. Certainly $DF_{\mathfrak{N}} \subseteq \chi$, since $\mathfrak{N} \models$ PA . To show $\chi \subseteq DF_{\mathfrak{N}}$, let $X \in \chi$. Then by using Gödel's β function, for example, we see by induction that there is a code for $\{x \in X: x <^{\mathfrak{N}} n\}$ for each $n \in N$ (via some uniform coding). By the theorem, $X \in DF_{\mathfrak{N}}$, as desired.

PROOF OF THEOREM. The proof uses the method of Keisler and Kunen [3] and the following results which can be found as indicated in Barwise [1]: Barwise Compactness Theorem (III5.6); Barwise (Extended) Completeness Theorem (III5.7); (Schlipf) $\mathfrak{N}$ is recursively saturated if and only if $HYP_{\mathfrak{N}}$, the least admissible set with $\mathfrak{N}$ as a set of urelements, has ordinal ω (IV5.3); and for recursively saturated $\mathfrak{N}$, $DF_{\mathfrak{N}} = \{X \subseteq N: X \in HYP_{\mathfrak{N}}\}$ (II7.2).

Let $\langle S_\alpha: \alpha < \omega_1 \rangle$ be as in the definition of $\diamond_{\omega_1}$. We can assume that $\mathfrak{M}$ has universe ω. We construct (as in Keisler [3]) an elementary end extension chain of recursively saturated models

$$\mathfrak{M} = \mathfrak{M}_0 \preccurlyeq_e \mathfrak{M}_1 \preccurlyeq_e \cdots \preccurlyeq_e \mathfrak{M}_\alpha \preccurlyeq_e \cdots \qquad (\alpha < \omega_1),$$

where $M_\alpha = \alpha$ for limit ordinals α, by induction on α. Unless $\alpha = \gamma + 1$ for some limit ordinal γ, we set $\mathfrak{M}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{M}_\beta$, a process which preserves elementarity, end extensions, and recursive saturation. If $\alpha = \gamma + 1$ for some limit γ, and S_γ is definable in $\mathfrak{M}_\gamma$, let L be a language containing constant symbols $\bar{n}$ for all $n \in \mathfrak{M}_\gamma$ and $\bar{m}$ for all $m \in \omega$, in an effective way, along with new constant symbols c and d, binary function symbols $+$ and $\cdot$, and a binary relation symbol $\in$. Consider the following theory T formulated in the fragment of $L_{\infty\omega}$ given by $HYP_{\mathfrak{M}_\gamma}$:

KPU^+ (admissible set theory with a set of all urelements);

The first-order elementary diagram of $\mathfrak{M}$ (with urelement variables);

$\{\bar{n} < c: n \in M_\gamma\}$;

$\bigwedge_{n \in M_\gamma} \forall p(p < \bar{n} \to \bigvee_{r \in M_\gamma}(p = \bar{r}))$;

"d is a finite ordinal";

$\{\bar{m} \in d: m \in \omega\}$.

T is obviously Σ_1 on $HYP_{\mathfrak{M}_\gamma}$. Also, any $HYP_{\mathfrak{M}_\gamma}$-finite subset of T contains only finitely many sentences of the form "$d \in \bar{m}$" for $m \in \omega$, and thus has a model since every model of PA has an elementary end extension and every model has a HYP. By Barwise compactness, T has a countable model; but its well-founded part, which has ordinal ω, is also admissible by the Truncation Lemma (see Barwise [1, II8.4]). Thus its urelement structure is recursively saturated: call it $\mathfrak{M}_\alpha$. Without loss of generality, $M_\alpha = \gamma + \omega$ and $\mathfrak{M}_\gamma \prec_e \mathfrak{M}_\alpha$.

The remaining case occurs when $\alpha = \gamma + 1$, γ a limit, and S_γ is not definable in $\mathfrak{M}_\gamma$. In that case we let L and T be as before but we want to take $\mathfrak{M}_\alpha$ to be a model of T omitting the following set Σ of formulas:

$$\{``\bar{n}Ex" : n \in S_\gamma\} \cup \{\neg ``\bar{n}Ex" : n \notin S_\gamma\},$$

where by "$\bar{n}Ex$" we mean "$\bar{n}$ is in the set coded by x." We choose our coding so that if $A \subsetneq B \subsetneq M_\gamma$ and A and B have codes in $\mathfrak{M}_\gamma$, then the code for A is less than the code for B. It suffices to show that T locally omits Σ. Suppose not; say

$T \models \forall x(\varphi(x) \rightarrow ``\bar{n}Ex")$ for $n \in S_\gamma$, and

$T \models \forall x(\varphi(x) \rightarrow \neg ``\bar{n}Ex")$ for $n \notin S_\gamma$,

where $\varphi(x)$ is in the fragment of $L_{\infty\omega}$ given by $HYP_{\mathfrak{M}_\gamma}$ and $\varphi(x)$ is consistent with T. Then

$$n \in S_\gamma \Leftrightarrow T \models \forall x(\varphi(x) \rightarrow ``\bar{n}Ex")$$

and

$$n \notin S_\gamma \Leftrightarrow T \models \forall x(\varphi(x) \rightarrow \neg ``\bar{n}Ex").$$

Thus S is both Σ_1 and Π_1 on $HYP_{\mathfrak{M}_\gamma}$ by Barwise (extended) completeness. Since $S \subseteq M_\gamma \in HYP_{\mathfrak{M}_\gamma}$, $S \in HYP_{\mathfrak{M}_\gamma}$ by Δ_1-separation. Therefore (by [1, II7.2]), $S_\gamma \in DF_{\mathfrak{M}_\gamma}$, a contradiction. So Σ is locally omitted by T, and we can take $\mathfrak{M}_\alpha$ to be a model of T as before ($M_\alpha = \gamma + \omega, \mathfrak{M}_\gamma \prec_e \mathfrak{M}_\alpha$), except that here $\mathfrak{M}_\alpha$ also omits Σ.

Now that the chain is constructed, we check to see that $\mathfrak{N} = \bigcup_{\alpha<\omega_1}\mathfrak{M}_\alpha$ has the desired properties. $\mathfrak{N}$ is recursively saturated because it is the union of an elementary chain of recursively saturated models. Let $S \subseteq N$ such that for each $n \in N$, $\{x \in S: x <^{\mathfrak{N}} n\}$ is definable with parameters. $\{\gamma \in \omega_1 : (\mathfrak{M}_\gamma, S \cap \mathfrak{M}_\gamma) \prec (\mathfrak{N}, S)\}$ is easily seen to be closed and unbounded in ω_1, as is $\{\gamma \in \omega_1 : \gamma \text{ is a limit}\}$. So if

$$C = \{\gamma \in \omega_1 : \gamma \text{ is a limit and } (\mathfrak{M}_\gamma, S \cap \mathfrak{M}_\gamma) \prec (\mathfrak{N}, S)\},$$

then C is also c.u.b. Pick $\gamma \in C$ such that $S \cap \gamma = S_\gamma$, using $\diamondsuit_{\omega_1}$. Then $(\mathfrak{M}_\gamma, S_\gamma) \prec (\mathfrak{N}, S)$. Pick any $n \in M_{\gamma+1} - M_\gamma$. Then $\{x \in S: x <^{\mathfrak{N}} n\}$ is definable in $\mathfrak{N}$, so it has a code r in $\mathfrak{N}$. By our choice of coding, $r <^{\mathfrak{N}}$ (the code for $\{x \in N: x <^{\mathfrak{N}} n\}$). But the latter is in $M_{\gamma+1}$, so since $\mathfrak{M}_{\gamma+1} \subset_e \mathfrak{N}$, $r \in M_{\gamma+1}$. By construction of $\mathfrak{M}_{\gamma+1}$, S_γ must be definable in $\mathfrak{M}_\gamma$. This same definition puts S in $DF_{\mathfrak{N}}$, since $(\mathfrak{M}_\gamma, S_\gamma) \prec (\mathfrak{N}, S)$.

Remark. The following "effective version" of part of Theorem B of Keisler [3] can be proved essentially as the above theorem is proved:

Assume $\diamondsuit_{\omega_1}$. Let $\mathfrak{A}$ be a countable tree-like model which satisfies the collection scheme, and suppose $\mathfrak{A}$ is recursively saturated. Then $\mathfrak{A}$ has a recursively saturated end elementary extension $\mathfrak{B}$ such that $\mathfrak{B}$ is an ω_1-like model and every branch of $\mathfrak{B}$ is definable in $\mathfrak{B}$.

Since (as Keisler shows) models of ZF and ZF minus infinity can be viewed as tree-like models, there is an obvious analogue of the theorem (proved above for PA) for models of these theories.

[**Added in proof October 19, 1976.** S. Shelah has informed the author of an absoluteness result which implies that $\diamond_{\omega_1}$ can be eliminated from the hypothesis of the result here as well as in the Keisler-Kunen result. (See his AMS Notices of October 1975 and February 1976.)]

References

1. K. J. Barwise, *Admissible sets and structures*, Springer-Verlag, Heidelberg, 1975.

2. K. J. Barwise and J. Schlipf, *On recursively saturated models of arithmetic*, Model Theory and Algebra, Lecture Notes in Math., vol. 498, Springer-Verlag, Berlin, 1976, 42–55.

3. H. J. Keisler, *Models with tree structures*, Proc. Tarski Sympos., Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, R.I., 1974, pp. 331–348.

4. J. Schlipf, *Some hyperelementary aspects of model theory*, Doctoral Dissertation, University of Wisconsin, Madison, Wis., 1975.

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