SIMPLE GOING DOWN IN PI RINGS

PHILLIP LESTMANN

Abstract. In this paper we prove two generalizations of a theorem which McAdam proved for commutative rings. Theorem 1 states that if \( R \subseteq S \) is a central integral extension of PI rings, then going down for prime ideals holds between \( R \) and \( S \) if and only if going down holds in \( R \subseteq R[s] \) for each \( s \in S \). Theorem 2 gives the analogous result for going down in \( C \subseteq R \) where \( C \) is a central subring of the PI ring \( R \). As a corollary we obtain a result of Schelter generalizing Krull's theorem on going down for integral extensions of integrally-closed subrings.

If \( R \) is a ring satisfying a polynomial identity, we shall say that \( R \) is a PI ring. A pair of rings \( R \subseteq S \) is said to have going down (GD) if for every pair of prime ideals \( P \subseteq Q \) of \( R \) and every prime ideal \( Q \) of \( S \) with \( Q \cap R = P \), there exists a prime ideal \( Q_1 \) of \( S \) so that \( Q_1 \subseteq Q \) and \( Q_1 \cap R = P \). The pair \( R \subseteq S \) is said to have simple going down (SGD) if \( R \subseteq R[t] \) has GD for each \( t \in S \).

If \( R \subseteq S \) is a pair of rings, \( S \) is said to be integral over \( R \) if each element \( s \) of \( S \) satisfies an equation of the form \( s^n + r_1 s^{n-1} + \cdots + r_{n-1} s + r_n = 0 \) where \( r_i \in R \), \( 1 \leq i \leq n \). McAdam has shown [1] the following: If \( R \subseteq T \) is a pair of commutative rings and \( T \) is integral over \( R \), then \( R \subseteq T \) has GD if and only if \( R \subseteq T \) has SGD. To extend this idea we first need another definition.

Definition. For any rings \( R \subseteq S \) let \( SR = \{ s \in S \mid sr = rs \text{ for each } r \in R \} \). We say that \( R \subseteq S \) is an extension if \( S = RSR \). The extension is said to be central if \( S = RZ(S) \) where \( Z(S) \) is the center of \( S \).

In this paper we generalize McAdam's result in Theorems 1 and 2 as follows:

Theorem 1. Let \( R \subseteq S \) be a central integral extension of PI rings. Then \( R \subseteq S \) has GD if and only if \( R \subseteq S \) has SGD.
Theorem 2. Let $R$ be a PI ring integral over the central subring $C$. Then $C \subseteq R$ has GD if and only if $C \subseteq R$ has SGD.

As a corollary to Theorem 2 we get the generalization of Krull's going down theorem for integral extensions of integrally-closed subrings which was proved by Schelter [4].

We begin with the following definition by way of reminder.

Definition. Let $R \subseteq S$ be a pair of rings.

1. If for each prime $P$ of $R$ there exists a prime $Q$ of $S$ so that $Q \cap R = P$, then $R \subseteq S$ is said to have lying over (LO).

2. If for any pair of primes $P \subseteq R$ of $R$ and prime $Q$ of $S$ with $Q \cap R = P$ there is a prime $Q_1$ of $S$ with $Q \subseteq Q_1$ and $Q_1 \cap R = R$, then $R \subseteq S$ is said to have going up (GU).

3. If $P$ any prime of $R$ and $Q_1$ and $Q_2$ primes of $S$ with $Q_1 \cap R = Q_2 \cap R = P$ implies that $Q_1 \subseteq Q_2$ and $Q_2 \subseteq Q_1$, then $R \subseteq S$ is said to have incomparability (INC).

Lemma 1. Let $R$ be any ring and let $B$ be a commutative subring of $R$. Assume $R$ is integral over $B$. If $P$ is any prime ideal of $B$ and $z \in P$, then $zR \cap B \subseteq P$.

Proof. Take $z \in P$ and suppose $zr \in B$ for some $r \in R$. $R$ is integral over $B$; so we have an equation $r^n + b_1 r^{n-1} + \cdots + b_n = 0$ for some $b_i \in B$, $1 \leq i \leq n$. Since $zr \in B$ and $B$ is commutative, $zr = z^2r$. Hence, we may write

$$0 = z^n(r^n + b_1 r^{n-1} + \cdots + b_n) = (zr)^n + b_1 z(zr)^{n-1} + \cdots + b_n z^n.$$

Thus $(zr)^n \in P$, and so $zr \in P$ since $P$ is a prime ideal of the commutative ring $B$.

The following lemma is Theorem 1 of [3]:

Lemma 2. If $R \subseteq S$ is an integral extension and $R$ satisfies a polynomial identity, then $R \subseteq S$ has GU, INC, and LO.

The next lemma is not new, but it is added here for completeness.

Lemma 3. If $R$ is any ring and $P$ is a minimal prime ideal of $R$, then $P \cap Z(R)$ consists of zero-divisors of $R$.

Proof. Let $A_0 = \{ r \in R | r$ is regular in $R$ (i.e., $r$ has no nontrivial right or left annihilator in $R$)$\}$, and set $A = A_0 \cap Z(R)$. If $A_0 \cap Z(R) = \emptyset$, we are done trivially. Let $B = R - P$. Write $AB = \{ ab | a \in A, b \in B \}$. Note that $0 \notin AB$ since $0 \notin B$ and $A$ consists of regular elements. Let $Q$ be an ideal of $R$ maximal with respect to the property $Q \cap AB = \emptyset$. Claim. $Q$ is a prime

3 Schelter has issued a correction to his paper which generalizes the definition of integral. He defines an element $s$ of an extension of a ring $R$ to be integral over $R$ if $s$ satisfies a monic polynomial in $R \subseteq C[x]$, the free product of $R$ and $C[x]$, where $C$ is the center of $R$. With a slight change in our proof of Lemma 1, the results of this paper hold for this more general (and useful) definition of integral.
ideal of \( R \). If not, there exist ideals \( I, J \) of \( R \) with \( Q \subseteq I, Q \subseteq J \), and \( IJ \subseteq Q \). By maximality of \( Q \) we may take \( x \in I \cap AB \) and \( y \in J \cap AB \). We have \( x, y \in R - Q \) and \( xRy \subseteq Q \). Since \( x, y \in AB \), we may set \( x = a_1b_1 \) and \( y = a_2b_2 \), where \( a_1, a_2 \in A, b_1, b_2 \in B \). Since \( b_1, b_2 \notin P \), there is an \( r \in R \) so that \( b_1rb_2 \notin P \). But \( xry = a_1b_1a_2b_2 = (a_1a_2)(b_1rb_2) \in AB \cap Q \). Contradiction. So \( Q \) is a prime ideal of \( R \).

Suppose \( Q \not\subseteq P \). Take \( b \in Q - P \) and any \( a \in A \). This gives \( ab \in Q \cap AB \). So we must have \( Q \subseteq P \). This implies that \( Q = P \) since \( P \) is a minimal prime. Now suppose \( Q \cap A \neq \emptyset \) and take \( a \in Q \cap A \). If \( b \in B \) we again have the contradiction \( ab \in Q \cap AB \). Therefore \( \emptyset = Q \cap A = P \cap A \), and we are done.

The next lemma was proved by McAdam for \( R \subseteq S \) both commutative in [1].

**Lemma 4.** Let \( R \subseteq S \) be any extension of \( \pi \)-rings, and let \( P \subseteq P_1 \) be prime ideals of \( R \). Suppose \( Q_1 \) is a prime of \( S \) with the property that \( Q_1 \cap R = R_1 \). Set \( \Gamma = \left\{ Q_a \mid Q_a \text{ is a prime ideal of } S, Q_a \cap R = P \right\} \), and define \( I = \bigcap \{ Q_a \mid Q_a \in \Gamma \} \). We assume \( \Gamma \neq \emptyset \). Then there is a prime \( Q \) of \( S \) with \( Q \subseteq Q_1 \) and \( Q \cap R = P \) if and only if \( I \subseteq Q_1 \).

**Proof.** (\( \Rightarrow \)) Obvious.

(\( \Leftarrow \)) Suppose \( I \subseteq Q_1 \). Then \( P \subseteq I \subseteq Q_1 \). Let \( Q \) be a prime ideal of \( S \) in \( Q_1 \) minimal over \( P \). If \( Q \cap R \neq P \), then \( \overline{Q} \cap \overline{R} \neq \emptyset \) in \( \overline{S} = S/I \). Now \( \overline{R} \subseteq \overline{S} \) is still an extension of \( \pi \)-rings. Since \( I \cap R = P, \overline{R} \) is a prime \( \pi \)-ring. So there is a \( \bar{q} \in Z(\overline{S}) \cap \overline{Q} \) with \( \bar{q} \neq \emptyset \) [5]. Since \( Z(\overline{R}) \subseteq Z(\overline{S}) \) and \( \overline{Q} \cap Z(\overline{S}) \) consists of zero-divisors by Lemma 3, we can find a \( i \neq 0 \) in \( \overline{S} \) so that \( \bar{q}i = \emptyset \). However, \( i \neq 0 \) gives a \( \overline{Q}_a \) for which \( i \notin \overline{Q}_a \). This forces \( \bar{q} \in \overline{Q}_a \). If not, \( \overline{S}/\overline{Q}_a \) is a prime \( \pi \)-ring with a nonzero central element which is a zero-divisor. Thus \( \bar{q} \) is in \( \overline{Q}_a \cap \overline{R} = \emptyset \) since \( Q_a \cap R = P \subseteq I \). This contradiction shows that \( \overline{Q} \cap \overline{R} = \emptyset \); i.e., \( Q \cap R = I \cap R = P \).

We now proceed to prove Theorems 1 and 2.

**Theorem 1.** Let \( R \subseteq S \) be a central integral extension of \( \pi \)-rings. Then \( R \subseteq S \) has GD if and only if \( R \subseteq S \) has SGD.

**Proof.** (\( \Rightarrow \)) Suppose \( R \subseteq S \) has GD, and let \( t \) be any element of \( S \). Let \( P \subseteq P_1 \) be two primes of \( R \) and \( Q_1 \) a prime of \( R[t] \) such that \( Q_1 \cap R = R_1 \). Since \( R[t] \subseteq S \) is a central integral extension, there is a prime \( Q'_1 \) of \( S \) with \( Q'_1 \cap R[t] = Q_1 \) by Lemma 2. Note that \( Q'_1 \cap R = P \); so there is a prime \( Q' \) of \( S \) satisfying \( Q' \subseteq Q'_1 \) and \( Q' \cap R = P \) by GD in \( R \subseteq S \). Let \( Q = Q' \cap R[t] \). Again, \( R[t] \subseteq S \) being a central extension gives that \( Q \) is a prime ideal of \( R[t] \). Furthermore, \( Q \subseteq Q_1 \) and \( Q \cap R = P \). Thus \( R \subseteq R[t] \) has GD for every \( t \in S \).

(\( \Leftarrow \)) Assume that \( R \subseteq S \) has SGD. Suppose there are two primes \( P \subseteq P_1 \) of \( R \) and a prime \( Q'_1 \) of \( S \) so that \( Q'_1 \cap R = P_1 \). Define \( I \) as in Lemma 4 using \( W = \{ Q_a \mid Q_a \text{ is a prime ideal of } S, Q_a \cap R = P \} \). Note that \( R \subseteq S \) has LO
by Lemma 2; hence, \( W \neq \emptyset \). By Lemma 4 we will be done if we can show that \( I \subseteq Q' \). If this is not true, pick \( t \in I - Q' \) and consider \( R[t] \). Let \( Q'_1 = Q' \cap R[t] \) which is prime in \( R[t] \) as above. By GD in \( R[t] \) there is a prime ideal \( Q \) of \( R[t] \) such that \( Q \subseteq Q'_1 \) and \( Q \cap R = P \). As before \( R[t] \subseteq S \) has LO; so there exists a prime \( Q' \) of \( S \) with \( Q' \cap R[t] = Q \). But then \( Q' \cap R = P \) gives \( I \subseteq Q' \) which says \( t \in Q' \cap R[t] = Q \subseteq Q'_1 = Q'_1 \cap R[t] \). This contradicts our choice of \( t \). Therefore, \( I \subseteq Q'_1 \) as required.

**Theorem 2.** Let \( R \) be a PI ring integral over a central subring \( C \). Then \( C \subseteq R \) has GD if and only if \( C \subseteq R \) has SGD.

**Proof.** (\( \Rightarrow \)) Suppose \( C \subseteq R \) has GD, and let \( t \) be any element of \( R \). Set \( B = C[t] \). Let \( P \subseteq R \) be two primes of \( C \) such that there is a prime \( Q_1 \) of \( B \) lying over \( P \) (i.e., \( P = Q_1 \cap C \)). Take \( q_1 \) to be an ideal of \( R \) maximal with respect to \( q_1 \cap B \subseteq Q_1 \). It is easy to see that \( q_1 \) is a prime ideal of \( R \). It is also true that \( q_1 \cap C = R \). For \( q_1 \cap C \subseteq (q_1 \cap B) \cap C \subseteq Q_1 \cap C = R \). If \( q_1 \cap C \subseteq P \), there is a \( z \in P - q_1 \cap C \). Consider \( R = R/q_1 \). Applying Lemma 1 to the rings \( B \subseteq R \), we find that \( zR \cap B \subseteq Q_1 \) which translates to \( (zR + q_1) \cap (B + q_1) \subseteq Q_1 + q_1 \) in \( R \). Taking the intersection of both sides with \( B \), we get \( (zR + q_1) \cap B \subseteq (Q_1 + q_1) \cap B = Q_1 \). But the ideal \( (z, q_1) \) is properly larger than \( q_1 \), contradicting the maximality of \( q_1 \). Thus \( q_1 \cap C = R \).

By hypothesis there is a prime ideal \( q \) of \( R \) such that \( q \subseteq q_1 \) and \( q \cap C = P \). Let \( S = C - P \) and \( T = B - Q_1 \). \( P \) and \( Q_1 \) are prime ideals of \( C \) and \( B \), respectively. Hence, \( ST = \{st \mid s \in S, t \in T \} \) is a multiplicatively closed subset of \( B \). Note that \( ST \cap (q \cap B) = \emptyset \) because elements of \( S \) are regular mod \( q \) in \( R \), hence in \( B \), and \( q \cap q_1 \) gives \( q \cap B \subseteq q_1 \cap B \subseteq Q_1 \) so that \( (q \cap B) \cap T = \emptyset \). Let \( Q \) be an ideal of \( B \) containing \( q \cap B \) and maximal with respect to \( Q \cap ST = \emptyset \). We note three things about \( Q \):

1. \( Q \) is a prime ideal of \( B \).
2. \( Q \cap C = P \). For \( Q \subseteq q \cap B \) implies \( Q \cap C \supseteq (q \cap B) \cap C = P \). If \( Q \cap C \supseteq P \), take any \( s \in (Q \cap C) - P \) and any \( t \in T \). Then \( s \in S \) and \( st \in Q \cap ST \). Contradiction.
3. \( Q \subseteq Q_1 \). If not, take \( t \in Q - Q_1 \) and any \( s \in S \). Then \( t \in T \) and \( st \in Q \cap ST \). Contradiction.

Therefore \( C \subseteq B = C[t] \) has GD.

(\( \Leftarrow \)) Suppose \( C \subseteq R \) has SGD. Let \( P \subseteq R \) be two primes of \( C \) and \( q_1 \) a prime of \( R \) lying over \( P \) in \( C \). Let \( W = \{q_\alpha \mid q_\alpha \text{ is prime in } R \text{ and } q_\alpha \cap C = P \} \). \( W \neq \emptyset \) since \( C \subseteq R \) has LO by Lemma 2. Set \( I = \cap \{q_\alpha \mid W \} \). By Lemma 4 it will be enough to show that \( I \subseteq q_1 \). If not, choose \( t \in I \) so that \( t \) is regular mod \( q_1 \) \([2, p. 48]\). Let \( B = C[t] \), and let \( Q_1 \) be a prime ideal of \( B \) with \( Q_1 \supseteq q_1 \cap B \) and \( Q_1 \cap C = P \). (Just enlarge \( q_1 \cap B \), if necessary, to an ideal of \( B \) maximal with respect to lying over \( P \).) Apply the SGD hypothesis to find a prime ideal \( Q \) of \( B \) so that \( Q \subseteq Q_1 \) and \( Q \cap C = P \). If we now take \( q \) to be an ideal of \( R \) maximal with respect to \( q \cap B \subseteq Q \), the same argument of the first part of this proof shows that \( q \) is prime in \( R \) and \( q \cap C = P \). So
q = q_a \in W; \text{ whence } I \subseteq q, \text{ and } t \subseteq q. \text{ Thus } t \subseteq q \cap B \subseteq Q \subseteq Q_1 \text{ implies } t \subseteq Q_1. \text{ Now } Q_1 \text{ is a prime minimal over the ideal } q_1 \cap B \text{ in } B. \text{ Otherwise, there is a prime } Q_0 \text{ of } B \text{ such that } q_1 \cap B \subseteq Q_0 \subseteq Q_1. \text{ This would give } P_i = (q_1 \cap B) \cap C \subseteq Q_0 \cap C \subseteq Q_1 \cap C = P_i, \text{ contradicting INC in } C \subseteq B. \text{ But } Q_1 \text{ minimal over } q_1 \cap B \text{ implies that } Q_1 \text{ consists of elements which are zero-divisors mod } q_1 \cap B \text{ by Lemma 3. So there exists } x \in B - q_1 \cap B \text{ so that } xt \in q_1 \cap B. \text{ This contradicts the fact that } t \text{ is regular mod } q_1 \text{ in } R. \text{ Therefore } I \subseteq q_1, \text{ and } C \subseteq R \text{ has GD.}

**Corollary (Schelter).** If \( R \) is a prime PI ring integral over an integrally-closed central subring \( A \), then \( A \subseteq R \) has GD.

**Proof.** By Theorem 2, \( A \subseteq R \) has GD if and only if \( A \subseteq R \) has SGD. If \( t \in R \), then \( A[t] \) is a commutative subring of \( R \) with no zero-divisors in \( A \). The commutative going down theorem (see e.g. [6]) may be applied to see that \( A \subseteq A[t] \) has GD.

**References**

1. Stephen McAdam, Private communication.