PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 63, Number 1, March 1977

SIMPLE GOING DOWN IN PI RINGS

PHILLIP LESTMANN[1]

ABSTRACT. In this paper we prove two generalizations of a theorem which McAdam proved for commutative rings. Theorem 1 states that if $R \subset S$ is a central integral extension of PI rings, then going down for prime ideals holds between R and S if and only if going down holds in $R \subset R[s]$ for each $s \in S$. Theorem 2 gives the analogous result for going down in $C \subset R$ where C is a central subring of the PI ring R. As a corollary we obtain a result of Schelter generalizing Krull's theorem on going down for integral extensions of integrally-closed subrings.

If R is a ring satisfying a polynomial identity, we shall say that R is a PI *ring*. A pair of rings $R \subset S$ is said to have *going down* (GD) if for every pair of prime ideals $P_1 \subset P$ of R and every prime ideal Q of S with $Q \cap R = P$, there exists a prime ideal Q_1 of S so that $Q_1 \subset Q$ and $Q_1 \cap R = P_1$. The pair $R \subset S$ is said to have *simple going down* (SGD) if $R \subset R[t]$ has GD for each $t \in S$.

If $R \subset S$ is a pair of rings, S is said to be *integral* over R if each element s of S satisfies an equation of the form $s^n + r_1 s^{n-1} + \cdots + r_{n-1} s + r_n = 0$ where $r_i \in R$, $1 \leqslant i \leqslant n$. McAdam has shown [1] the following: If $R \subset T$ is a pair of commutative rings and T is integral over R, then $R \subset T$ has GD if and only if $R \subset T$ has SGD.[2] To extend this idea we first need another definition.

DEFINITION. For any rings $R \subset S$ let $S^R = \{s \in S \mid sr = rs$ for each $r \in R\}$. We say that $R \subset S$ is an *extension* if $S = RS^R$. The extension is said to be *central* if $S = RZ(S)$ where $Z(S) =$ the center of S.

In this paper we generalize McAdam's result in Theorems 1 and 2 as follows:

THEOREM 1. *Let $R \subset S$ be a central integral extension of* PI *rings. Then $R \subset S$ has* GD *if and only if $R \subset S$ has* SGD.

Received by the editors August 6, 1976.

AMS (MOS) subject classifications (1970). Primary 16A38, 16A66.

Key words and phrases. PI ring, going down, simple going down, integral, extension, going up, lying over, incomparability.

[1] This paper is part of the author's thesis for the Ph.D. degree at the University of Southern California.

[2] The author is indebted to the referee for pointing out that this result has appeared as Proposition 2 in McAdam, *Going down and open extensions*, Canad. J. Math. **27** (1975), 111–114.

THEOREM 2. *Let R be a* PI *ring integral over the central subring C. Then $C \subset R$ has* GD *if and only if $C \subset R$ has* SGD.

As a corollary to Theorem 2 we get the generalization of Krull's going down theorem for integral extensions of integrally-closed subrings which was proved by Schelter [4].

We begin with the following definition by way of reminder.

DEFINITION. Let $R \subset S$ be a pair of rings.

(1) If for each prime P of R there exists a prime Q of S so that $Q \cap R = P$, then $R \subset S$ is said to have *lying over* (LO).

(2) If for any pair of primes $P \subset P_1$ of R and prime Q of S with $Q \cap R = P$ there is a prime Q_1 of S with $Q \subset Q_1$ and $Q_1 \cap R = P_1$, then $R \subset S$ is said to have *going up* (GU).

(3) If P any prime of R and Q_1 and Q_2 primes of S with $Q_1 \cap R = Q_2 \cap R = P$ implies that $Q_1 \nsubseteq Q_2$ and $Q_2 \nsubseteq Q_1$, then $R \subset S$ is said to have *incomparability* (INC).

LEMMA 1. *Let R be any ring and let B be a commutative subring of R. Assume R is integral over B. If P is any prime ideal of B and $z \in P$, then $zR \cap B \subseteq P$.*

PROOF. Take $z \in P$ and suppose $zr \in B$ for some $r \in R$. R is integral over B; so we have an equation $r^n + b_1 r^{n-1} + \cdots + b_n = 0$ for some $b_i \in B$, $1 \leqslant i \leqslant n$. Since $zr \in B$ and B is commutative, $zrz = z^2 r$. Hence, we may write

$$0 = z^n(r^n + b_1 r^{n-1} + \cdots + b_n) = (zr)^n + b_1 z(zr)^{n-1} + \cdots + b_n z^n.$$

Thus $(zr)^n \in P$, and so $zr \in P$ since P is a prime ideal of the commutative ring B.

The following lemma is Theorem 1 of [3]:[3]

LEMMA 2. *If $R \subset S$ is an integral extension and R satisfies a polynomial identity, then $R \subset S$ has* GU, INC, *and* LO.

The next lemma is not new, but it is added here for completeness.

LEMMA 3. *If R is any ring and P is a minimal prime ideal of R, then $P \cap Z(R)$ consists of zero-divisors of R.*

PROOF. Let $A_0 = \{r \in R | r$ is regular in R (i.e., r has no nontrivial right or left annihilator in $R)\}$, and set $A = A_0 \cap Z(R)$. If $A_0 \cap Z(R) = \varnothing$, we are done trivially. Let $B = R - P$. Write $AB = \{ab | a \in A, b \in B\}$. Note that $0 \notin AB$ since $0 \notin B$ and A consists of regular elements. Let Q be an ideal of R maximal with respect to the property $Q \cap AB = \varnothing$. *Claim.* Q is a prime

[3] Schelter has issued a correction to his paper which generalizes the definition of integral. He defines an element s of an extension of a ring R to be integral over R if s satisfies a monic polynomial in $R \overset{*}{\subset} C[x]$, the free product of R and $C[x]$, where C is the center of R. With a slight change in our proof of Lemma 1, the results of this paper hold for this more general (and useful) definition of integral.

ideal of R. If not, there exist ideals I, J of R with $Q \subsetneqq I$, $Q \subsetneqq J$, and $IJ \subseteq Q$. By maximality of Q we may take $x \in I \cap AB$ and $y \in J \cap AB$. We have $x, y \in R - Q$ and $xRy \subseteq Q$. Since $x, y \in AB$, we may set $x = a_1 b_1$ and $y = a_2 b_2$, where $a_1, a_2 \in A$, $b_1, b_2 \in B$. Since $b_1, b_2 \notin P$, there is an $r \in R$ so that $b_1 r b_2 \notin P$. But $xry = a_1 b_1 r a_2 b_2 = (a_1 a_2)(b_1 r b_2) \in AB \cap Q$. Contradiction. So Q is a prime ideal of R.

Suppose $Q \nsubseteq P$. Take $b \in Q - P$ and any $a \in A$. This gives $ab \in Q \cap AB$. So we must have $Q \subseteq P$. This implies that $Q = P$ since P is a minimal prime. Now suppose $Q \cap A \neq \varnothing$ and take $a \in Q \cap A$. If $b \in B$ we again have the contradiction $ab \in Q \cap AB$. Therefore $\varnothing = Q \cap A = P \cap A$, and we are done.

The next lemma was proved by McAdam for $R \subset S$ both commutative in [1].

LEMMA 4. *Let $R \subset S$ be any extension of* PI *rings, and let $P \subset P_1$ be prime ideals of R. Suppose Q_1 is a prime of S with the property that $Q_1 \cap R = P_1$. Set $W = \{Q_\alpha | Q_\alpha$ is a prime ideal of S, $Q_\alpha \cap R = P\}$, and define $I = \bigcap\{Q_\alpha | Q_\alpha \in W\}$. We assume $W \neq \varnothing$. Then there is a prime Q of S with $Q \subset Q_1$ and $Q \cap R = P$ if and only if $I \subset Q_1$.*

PROOF. ($\Rightarrow$) Obvious.

($\Leftarrow$) Suppose $I \subset Q_1$. Then $P \subset I \subset Q_1$. Let Q be a prime ideal of S in Q_1 minimal over P. If $Q \cap R \neq P$, then $\overline{Q} \cap \overline{R} \neq \overline{0}$ in $\overline{S} = S/I$. Now $\overline{R} \subset \overline{S}$ is still an extension of PI rings. Since $I \cap R = P$, $\overline{R}$ is a prime PI ring. So there is a $\overline{q} \in Z(\overline{R}) \cap \overline{Q}$ with $\overline{q} \neq \overline{0}$ [5]. Since $Z(\overline{R}) \subset Z(\overline{S})$ and $\overline{Q} \cap Z(\overline{S})$ consists of zero-divisors by Lemma 3, we can find a $\overline{t} \neq \overline{0}$ in $\overline{S}$ so that $\overline{qt} = \overline{0}$. However, $\overline{t} \neq \overline{0}$ gives a $\overline{Q}_\alpha$ for which $\overline{t} \notin \overline{Q}_\alpha$. This forces $\overline{q} \in \overline{Q}_\alpha$. If not, $\overline{S}/\overline{Q}_\alpha$ is a prime PI ring with a nonzero central element which is a zero-divisor. Thus $\overline{q}$ is in $\overline{Q}_\alpha \cap \overline{R} = \overline{0}$ since $Q_\alpha \cap R = P \subset I$. This contradiction shows that $\overline{Q} \cap \overline{R} = \overline{0}$; i.e., $Q \cap R = I \cap R = P$.

We now proceed to prove Theorems 1 and 2.

THEOREM 1. *Let $R \subset S$ be a central integral extension of* PI *rings. Then $R \subset S$ has* GD *if and only if $R \subset S$ has* SGD.

PROOF. ($\Rightarrow$) Suppose $R \subset S$ has GD, and let t be any element of S. Let $P \subset P_1$ be two primes of R and Q_1 a prime of $R[t]$ such that $Q_1 \cap R = P_1$. Since $R[t] \subset S$ is a central integral extension, there is a prime Q_1' of S with $Q_1' \cap R[t] = Q_1$ by Lemma 2. Note that $Q_1' \cap R = P$; so there is a prime Q' of S satisfying $Q' \subset Q_1'$ and $Q' \cap R = P$ by GD in $R \subset S$. Let $Q = Q' \cap R[t]$. Again, $R[t] \subset S$ being a central extension gives that Q is a prime ideal of $R[t]$. Furthermore, $Q \subset Q_1$ and $Q \cap R = P$. Thus $R \subset R[t]$ has GD for every $t \in S$.

($\Leftarrow$) Assume that $R \subset S$ has SGD. Suppose there are two primes $P \subset P_1$ of R and a prime Q_1' of S so that $Q_1' \cap R = P_1$. Define I as in Lemma 4 using $W = \{Q_\alpha | Q_\alpha$ is a prime ideal of S, $Q_\alpha \cap R = P\}$. Note that $R \subset S$ has LO

by Lemma 2; hence, $W \neq \varnothing$. By Lemma 4 we will be done if we can show that $I \subseteq Q_1'$. If this is not true, pick $t \in I - Q_1'$ and consider $R[t]$. Let $Q_1 = Q_1' \cap R[t]$ which is prime in $R[t]$ as above. By GD in $R[t]$ there is a prime ideal Q of $R[t]$ such that $Q \subset Q_1$ and $Q \cap R = P$. As before $R[t] \subset S$ has LO; so there exists a prime Q' of S with $Q' \cap R[t] = Q$. But then $Q' \cap R = P$ gives $I \subset Q'$ which says $t \in Q' \cap R[t] = Q \subset Q_1 = Q_1' \cap R[t]$. This contradicts our choice of t. Therefore, $I \subset Q_1'$ as required.

THEOREM 2. *Let R be a* PI *ring integral over a central subring C. Then $C \subset R$ has* GD *if and only if $C \subset R$ has* SGD.

PROOF. ($\Rightarrow$) Suppose $C \subset R$ has GD, and let t be any element of R. Set $B = C[t]$. Let $P \subset P_1$ be two primes of C such that there is a prime Q_1 of B lying over P_1 (i.e., $P_1 = Q_1 \cap C$). Take q_1 to be an ideal of R maximal with respect to $q_1 \cap B \subseteq Q_1$. It is easy to see that q_1 is a prime ideal of R. It is also true that $q_1 \cap C = P_1$. For $q_1 \cap C \subseteq (q_1 \cap B) \cap C \subseteq Q_1 \cap C = P_1$. If $q_1 \cap C \subsetneq P_1$, there is a $z \in P_1 - q_1 \cap C$. Consider $\overline{R} = R/q_1$. Applying Lemma 1 to the rings $\overline{B} \subset \overline{R}$, we find that $\overline{zR} \cap \overline{B} \subseteq \overline{Q}_1$ which translates to $(zR + q_1) \cap (B + q_1) \subseteq Q_1 + q_1$ in R. Taking the intersection of both sides with B, we get $(zR + q_1) \cap B \subseteq (Q_1 + q_1) \cap B = Q_1$. But the ideal (z, q_1) is properly larger than q_1, contradicting the maximality of q_1. Thus $q_1 \cap C = P_1$.

By hypothesis there is a prime ideal q of R such that $q \subset q_1$ and $q \cap C = P$. Let $S = C - P$ and $T = B - Q_1$. P and Q_1 are prime ideals of C and B, respectively. Hence, $ST = \{st \mid s \in S, t \in T\}$ is a multiplicatively closed subset of B. Note that $ST \cap (q \cap B) = \varnothing$ because elements of S are regular mod q in R, hence in B, and $q \subset q_1$ gives $q \cap B \subset q_1 \cap B \subseteq Q_1$ so that $(q \cap B) \cap T = \varnothing$. Let Q be an ideal of B containing $q \cap B$ and maximal with respect to $Q \cap ST = \varnothing$. We note three things about Q:

(1) Q is a prime ideal of B.

(2) $Q \cap C = P$. For $Q \supseteq q \cap B$ implies $Q \cap C \supseteq (q \cap B) \cap C = P$. If $Q \cap C \supsetneq P$, take any $s \in (Q \cap C) - P$ and any $t \in T$. Then $s \in S$ and $st \in Q \cap ST$. Contradiction.

(3) $Q \subset Q_1$. If not, take $t \in Q - Q_1$ and any $s \in S$. Then $t \in T$ and $st \in Q \cap ST$. Contradiction.

Therefore $C \subseteq B = C[t]$ has GD.

($\Leftarrow$) Suppose $C \subset R$ has SGD. Let $P \subset P_1$ be two primes of C and q_1 a prime of R lying over P_1 in C. Let $W = \{q_\alpha \mid q_\alpha$ is prime in R and $q_\alpha \cap C = P\}$. $W \neq \varnothing$ since $C \subset R$ has LO by Lemma 2. Set $I = \bigcap\{q_\alpha \mid q_\alpha \in W\}$. By Lemma 4 it will be enough to show that $I \subset q_1$. If not, choose $t \in I$ so that t is regular mod q_1 [2, p. 48]. Let $B = C[t]$, and let Q_1 be a prime ideal of B with $Q_1 \supseteq q_1 \cap B$ and $Q_1 \cap C = P_1$. (Just enlarge $q_1 \cap B$, if necessary, to an ideal of B maximal with respect to lying over P_1.) Apply the SGD hypothesis to find a prime ideal Q of B so that $Q \subset Q_1$ and $Q \cap C = P$. If we now take q to be an ideal of R maximal with respect to $q \cap B \subseteq Q$, the same argument of the first part of this proof shows that q is prime in R and $q \cap C = P$. So

$q = q_\alpha \in W$; whence $I \subset q$, and $t \in q$. Thus $t \in q \cap B \subseteq Q \subset Q_1$ implies $t \in Q_1$. Now Q_1 is a prime minimal over the ideal $q_1 \cap B$ in B. Otherwise, there is a prime Q_0 of B such that $q_1 \cap B \subseteq Q_0 \subset Q_1$. This would give $P_1' = (q_1 \cap B) \cap C \subseteq Q_0 \cap C \subseteq Q_1 \cap C = P_1'$, contradicting INC in $C \subseteq B$. But Q_1 minimal over $q_1 \cap B$ implies that Q_1 consists of elements which are zero-divisors mod $q_1 \cap B$ by Lemma 3. So there exists $x \in B - q_1 \cap B$ so that $xt \in q_1 \cap B$. This contradicts the fact that t is regular mod q_1 in R. Therefore $I \subset q_1$, and $C \subset R$ has GD.

COROLLARY (SCHELTER). *If R is a prime* PI *ring integral over an integrally-closed central subring A, then $A \subseteq R$ has* GD.

PROOF. By Theorem 2, $A \subseteq R$ has GD if and only if $A \subseteq R$ has SGD. If $t \in R$, then $A[t]$ is a commutative subring of R with no zero-divisors in A. The commutative going down theorem (see e.g. [6]) may be applied to see that $A \subseteq A[t]$ has GD.

REFERENCES

1. Stephen McAdam, Private communication.
2. Claudio Procesi, *Rings with polynomial identities*, Dekker, New York, 1973. MR **51** #3214.
3. William Schelter, *Integral extensions of rings satisfying a polynomial identity*, J. Algebra **40** (1976), 245–257.
4. ———, *Non-commutative affine* P.I. *rings are Catenary* (to appear).
5. L. Rowen, *Some results on the center of a ring with polynomial identity*, Bull. Amer. Math. Soc. **79** (1973), 219–223. MR **46** #9099.
6. I. Cohen and A. Seidenberg, *Prime ideals and integral dependence*, Bull. Amer. Math. Soc. **52** (1946), 252–261. MR **7**, 410.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CALIFORNIA 90007