SIMPLE GOING DOWN IN PI RINGS

PHILLIP LESTMANN

ABSTRACT. In this paper we prove two generalizations of a theorem which McAdam proved for commutative rings. Theorem 1 states that if $R \subset S$ is a central integral extension of PI rings, then going down for prime ideals holds between $R$ and $S$ if and only if going down holds in $R \subset R[t]$ for each $t \in S$. Theorem 2 gives the analogous result for going down in $C \subset R$ where $C$ is a central subring of the PI ring $R$. As a corollary we obtain a result of Schelter generalizing Krull’s theorem on going down for integral extensions of integrally-closed subrings.

If $R$ is a ring satisfying a polynomial identity, we shall say that $R$ is a PI ring. A pair of rings $R \subset S$ is said to have going down (GD) if for every pair of prime ideals $P \subset P$ of $R$ and every prime ideal $Q$ of $S$ with $Q \cap R = P$, there exists a prime ideal $Q_1$ of $S$ so that $Q_1 \subset Q$ and $Q_1 \cap R = P$. The pair $R \subset S$ is said to have simple going down (SGD) if $R \subset R[t]$ has GD for each $t \in S$.

If $R \subset S$ is a pair of rings, $S$ is said to be integral over $R$ if each element $s$ of $S$ satisfies an equation of the form $s^n + r_1 s^{n-1} + \cdots + r_{n-1} s + r_n = 0$ where $r_i \in R$, $1 \leq i \leq n$. McAdam has shown [1] the following: If $R \subset T$ is a pair of commutative rings and $T$ is integral over $R$, then $R \subset T$ has GD if and only if $R \subset T$ has SGD. To extend this idea we first need another definition.

Definition. For any rings $R \subset S$ let $S^R = \{s \in S | sr = rs \text{ for each } r \in R\}$. We say that $R \subset S$ is an extension if $S = RS^R$. The extension is said to be central if $S = RZ(S)$ where $Z(S)$ is the center of $S$.

In this paper we generalize McAdam’s result in Theorems 1 and 2 as follows:

**Theorem 1.** Let $R \subset S$ be a central integral extension of PI rings. Then $R \subset S$ has GD if and only if $R \subset S$ has SGD.

Received by the editors August 6, 1976.

**AMS (MOS) subject classifications (1970).** Primary 16A38, 16A66.

**Key words and phrases.** PI ring, going down, simple going down, integral extension, going up, lying over, incomparability.

1 This paper is part of the author’s thesis for the Ph.D. degree at the University of Southern California.

2 The author is indebted to the referee for pointing out that this result has appeared as Proposition 2 in McAdam, Going down and open extensions, Canad. J. Math. 27 (1975), 111–114.
Theorem 2. Let $R$ be a PI ring integral over the central subring $C$. Then $C \subseteq R$ has GD if and only if $C \subseteq R$ has SGD.

As a corollary to Theorem 2 we get the generalization of Krull’s going down theorem for integral extensions of integrally-closed subrings which was proved by Schelter [4].

We begin with the following definition by way of reminder.

Definition. Let $R \subseteq S$ be a pair of rings.

1. If for each prime $P$ of $R$ there exists a prime $Q$ of $S$ so that $Q \cap R = P$, then $R \subseteq S$ is said to have lying over (LO).

2. If for any pair of primes $P \subseteq R$ of $R$ and prime $Q$ of $S$ with $Q \cap R = P$ there is a prime $Q_1$ of $S$ with $Q \subseteq Q_1$ and $Q_1 \cap R = P$, then $R \subseteq S$ is said to have going up (GU).

3. If $P$ any prime of $R$ and $Q_1$ and $Q_2$ primes of $S$ with $Q_1 \cap R = Q_2 \cap R = P$ implies that $Q_1 \not< Q_2$ and $Q_2 \not< Q_1$, then $R \subseteq S$ is said to have incomparability (INC).

Lemma 1. Let $R$ be any ring and let $B$ be a commutative subring of $R$. Assume $R$ is integral over $B$. If $P$ is any prime ideal of $B$ and $z \in P$, then $zR \cap B \subseteq P$.

Proof. Take $z \in P$ and suppose $zr \in B$ for some $r \in R$. $R$ is integral over $B$; so we have an equation $r^n + b_1 r^{n-1} + \cdots + b_n = 0$ for some $b_i \in B$, $1 \leq i \leq n$. Since $zr \in B$ and $B$ is commutative, $zrz = z^2 r$. Hence, we may write

$$0 = z^n(r^n + b_1 r^{n-1} + \cdots + b_n) = (zr)^n + b_1 z(zr)^{n-1} + \cdots + b_n z^n.$$  

Thus $(zr)^n \in P$, and so $zr \in P$ since $P$ is a prime ideal of the commutative ring $B$.

The following lemma is Theorem 1 of [3]:

Lemma 2. If $R \subseteq S$ is an integral extension and $R$ satisfies a polynomial identity, then $R \subseteq S$ has GU, INC, and LO.

The next lemma is not new, but it is added here for completeness.

Lemma 3. If $R$ is any ring and $P$ is a minimal prime ideal of $R$, then $P \cap Z(R)$ consists of zero-divisors of $R$.

Proof. Let $A_0 = \{r \in R | r$ is regular in $R$ (i.e., $r$ has no nontrivial right or left annihilator in $R)\}$, and set $A = A_0 \cap Z(R)$. If $A_0 \cap Z(R) = \emptyset$, we are done trivially. Let $B = R - P$. Write $AB = \{ab | a \in A, b \in B\}$. Note that $0 \notin AB$ since $0 \notin B$ and $A$ consists of regular elements. Let $Q$ be an ideal of $R$ maximal with respect to the property $Q \cap AB = \emptyset$. Claim. $Q$ is a prime
ideal of \( R \). If not, there exist ideals \( I, J \) of \( R \) with \( Q \subseteq I, Q \subseteq J \), and \( IJ \subseteq Q \).

By maximality of \( Q \) we may take \( x \in I \cap AB \) and \( y \in J \cap AB \). We have \( x, y \in R - Q \) and \( xy \subseteq Q \). Since \( x, y \in AB \), we may set \( x = a_1b_1 \) and \( y = a_2b_2 \), where \( a_1, a_2 \subseteq A, b_1, b_2 \subseteq B \). Since \( b_1, b_2 \notin P \), there is an \( r \in R \) so that \( b_1rb_2 \notin P \). But \( xy = a_1b_1a_2b_2 = (a_1a_2)(b_1rb_2) \in AB \cap Q \). Contradiction. So \( Q \) is a prime ideal of \( R \).

Suppose \( Q \not\subseteq P \). Take \( b \in Q - P \) and any \( a \in A \). This gives \( ab \in Q \cap AB \). So we must have \( Q \subseteq P \). This implies that \( Q = P \) since \( P \) is a minimal prime. Now suppose \( Q \cap A \neq \emptyset \) and take \( a \in Q \cap A \). If \( b \in B \) we again have the contradiction \( ab \in Q \cap AB \). Therefore \( \emptyset = Q \cap A = P \cap A \), and we are done.

The next lemma was proved by McAdam for \( R \subseteq S \) both commutative in [1].

\textbf{Lemma 4.} Let \( R \subseteq S \) be any extension of PI rings, and let \( P \subseteq P \) be prime ideals of \( R \). Suppose \( Q_1 \) is a prime of \( S \) with the property that \( Q_1 \cap R = R \); Set \( W = \{ Q_\alpha | Q_\alpha \text{ is a prime ideal of } S, Q_\alpha \cap R = P \} \), and define \( I = \cap \{ Q_\alpha | Q_\alpha \in W \} \). We assume \( W \neq \emptyset \). Then there is a prime \( Q \) of \( S \) with \( Q \subseteq Q_1 \) and \( Q \cap R = P \) if and only if \( I \subseteq Q_1 \).

\textbf{Proof.} \( \Rightarrow \) Obvious.

\( \Leftarrow \) Suppose \( I \subseteq Q_1 \). Then \( P \subseteq I \subseteq Q_1 \). Let \( Q \) be a prime ideal of \( S \) in \( Q_1 \) minimal over \( P \). If \( Q \cap R \neq P \), then \( \overline{Q} \cap \overline{R} = \emptyset \) in \( \overline{S} = S/I \). Now \( \overline{R} \subseteq \overline{S} \) is still an extension of PI rings. Since \( I \cap R = P, \overline{R} \) is a prime PI ring. So there is a \( \overline{q} \in Z(\overline{R}) \cap \overline{Q} \) with \( \overline{q} \neq \emptyset \) [5]. Since \( Z(\overline{R}) \subseteq Z(\overline{S}) \) and \( \overline{Q} \cap Z(\overline{S}) \) consists of zero-divisors by Lemma 3, we can find a \( \overline{i} \neq \emptyset \) in \( \overline{S} \) so that \( \overline{qi} = \emptyset \). However, \( \overline{i} \neq \emptyset \) gives a \( \overline{Q}_\alpha \) for which \( i \notin \overline{Q}_\alpha \). This forces \( \overline{q} \in \overline{Q}_\alpha \). If not, \( S/\overline{Q}_\alpha \) is a prime PI ring with a nonzero central element which is a zero-divisor. Thus \( \overline{q} \) is in \( \overline{Q}_\alpha \cap \overline{R} = \emptyset \) since \( Q \subseteq R = P \subseteq I \). This contradiction shows that \( Q \cap R = \emptyset \); i.e., \( Q \cap R = I \cap R = P \).

We now proceed to prove Theorems 1 and 2.

\textbf{Theorem 1.} Let \( R \subseteq S \) be a central integral extension of PI rings. Then \( R \subseteq S \) has GD if and only if \( R \subseteq S \) has SGD.

\textbf{Proof.} \( \Rightarrow \) Suppose \( R \subseteq S \) has GD, and let \( i \) be any element of \( S \). Let \( P \subseteq P \) be two primes of \( R \) and \( Q_1 \) a prime of \( R[i] \) such that \( Q_1 \cap R = P \).

Since \( R[i] \subseteq S \) is a central integral extension, there is a prime \( Q'_1 \) of \( S \) with \( Q'_1 \cap R[i] = Q_1 \) by Lemma 2. Note that \( Q'_1 \cap R = P \); so there is a prime \( Q' \) of \( S \) satisfying \( Q' \subseteq Q'_1 \) and \( Q' \cap R = P \) by GD in \( R \subseteq S \). Let \( Q = Q' \cap R[i] \). Again, \( R[i] \subseteq S \) being a central extension gives that \( Q \) is a prime ideal of \( R[i] \). Furthermore, \( Q \subseteq Q_1 \) and \( Q \cap R = P \). Thus \( R \subseteq R[i] \) has GD for every \( i \in S \).

\( \Leftarrow \) Assume that \( R \subseteq S \) has SGD. Suppose there are two primes \( P \subseteq P \) of \( R \) and a prime \( Q'_1 \) of \( S \) so that \( Q'_1 \cap R = P \). Define \( I \) as in Lemma 4 using \( W = \{ Q_\alpha | Q_\alpha \text{ is a prime ideal of } S, Q_\alpha \cap R = P \} \). Note that \( R \subseteq S \) has LO.
by Lemma 2; hence, $W \neq \emptyset$. By Lemma 4 we will be done if we can show that $I \subseteq Q_1$. If this is not true, pick $t \in I - Q_1$ and consider $R[t]$. Let $Q_1' = Q_1 \cap R[t]$ which is prime in $R[t]$ as above. By GD in $R[t]$ there is a prime ideal $Q$ of $R[t]$ such that $Q \subset Q_1'$ and $Q \cap R = P$. As before $R[t] \subset S$ has LO; so there exists a prime $Q'$ of $S$ with $Q' \cap R[t] = Q$. But then $Q' \cap R = P$ gives $I \subseteq Q'$ which says $t \in Q' \cap R[t] = Q \subset Q_1' = Q_1$ \cap $R[t]$. This contradicts our choice of $t$. Therefore, $I \subset Q_1$ as required.

THEOREM 2. Let $R$ be a PI ring integral over a central subring $C$. Then $C \subset R$ has GD if and only if $C \subset R$ has SGD.

PROOF. ($\Rightarrow$) Suppose $C \subset R$ has GD, and let $t$ be any element of $R$. Set $B = C[t]$. Let $P \subset R$ be two primes of $C$ such that there is a prime $Q_1$ of $B$ lying over $P$ (i.e., $P = Q_1 \cap C$). Take $q_1$ to be an ideal of $R$ maximal with respect to $q_1 \cap B \subseteq Q_1$. It is easy to see that $q_1$ is a prime ideal of $R$. It is also true that $q_1 \cap C = \mathfrak{P}_1$. For $q_1 \cap C \subseteq (q_1 \cap B) \cap C \subseteq Q_1 \cap C = R$. If $q_1 \cap C \supseteq \mathfrak{P}_1$, there is a $z \in \mathfrak{P}_1 - q_1 \cap C$. Consider $\mathfrak{R} = R/q_1$. Applying Lemma 1 to the rings $\mathfrak{B} \subset \mathfrak{R}$, we find that $z\mathfrak{R} \cap \mathfrak{B} \subseteq \mathfrak{B}_1$ which translates to $(z\mathfrak{R} + q_1) \cap (\mathfrak{B} + q_1) \subseteq \mathfrak{Q}_1 + q_1$ in $R$. Taking the intersection of both sides with $B$, we get $(z\mathfrak{R} + q_1) \cap B \subseteq (Q_1 + q_1) \cap B = Q_1$. But the ideal $(z, q_1)$ is properly larger than $q_1$, contradicting the maximality of $q_1$. Thus $q_1 \cap C = \mathfrak{P}_1$.

By hypothesis there is a prime ideal $q$ of $R$ such that $q \cap q_1$ and $q \cap C = P$. Let $S = C - P$ and $T = B - Q_1$. $P$ and $Q_1$ are prime ideals of $C$ and $B$, respectively. Hence, $ST = \{st | s \in S, t \in T\}$ is a multiplicatively closed subset of $B$. Note that $ST \cap (q \cap B) = \emptyset$ because elements of $S$ are regular mod $q$ in $R$, hence in $B$, and $q \cap q_1$ gives $q \cap B \subset q_1 \cap B \subseteq Q_1$ so that $(q \cap B) \cap T = \emptyset$. Let $Q$ be an ideal of $B$ containing $q \cap B$ and maximal with respect to $Q \cap ST = \emptyset$. We note three things about $Q$:

1. $Q$ is a prime ideal of $B$.
2. $Q \cap C = P$. For $Q \supseteq q \cap B$ implies $Q \cap C \supseteq (q \cap B) \cap C = P$. If $Q \cap C \supseteq P$, take any $s \in (Q \cap C) - P$ and any $t \in T$. Then $s \in S$ and $st \in Q \cap ST$. Contradiction.
3. $Q \subset Q_1$. If not, take $t \in Q - Q_1$ and any $s \in S$. Then $t \in T$ and $st \in Q \cap ST$. Contradiction.

Therefore $C \subseteq B = C[t]$ has GD.

($\Leftarrow$) Suppose $C \subset R$ has SGD. Let $P \subset R$ be two primes of $C$ and $q_1$ a prime of $R$ lying over $P$ in $C$. Let $W = \{q_\alpha | q_\alpha \text{ is prime in } R \text{ and } q_\alpha \cap C = P\}$. $W \neq \emptyset$ since $C \subset R$ has LO by Lemma 2. Set $I = \cap \{q_\alpha | q_\alpha \in W\}$. By Lemma 4 it will be enough to show that $I \subset q_1$. If not, choose $t \in I$ so that $t$ is regular mod $q_1$ [2, p. 48]. Let $B = C[t]$, and let $Q_1$ be a prime ideal of $B$ with $Q_1 \supseteq q_1 \cap B$ and $Q_1 \cap C = \mathfrak{P}_1$. (Just enlarge $q_1 \cap B$, if necessary, to an ideal of $B$ maximal with respect to lying over $P$.) Apply the SGD hypothesis to find a prime ideal $Q$ of $B$ so that $Q \subset Q_1$ and $Q \cap C = P$. If we now take $q$ to be an ideal of $R$ maximal with respect to $q \cap B \subseteq Q$, the same argument of the first part of this proof shows that $q$ is prime in $R$ and $q \cap C = P$. So
\( q = q_a \in W \); whence \( I \subset q \), and \( t \notin q \). Thus \( t \in q \cap B \subseteq Q \subseteq Q_1 \) implies \( t \in Q_1 \). Now \( Q_1 \) is a prime minimal over the ideal \( q_1 \cap B \) in \( B \). Otherwise, there is a prime \( Q_0 \) of \( B \) such that \( q_1 \cap B \subseteq Q_0 \subseteq Q_1 \). This would give \( P_i = (q_1 \cap B) \cap C \subseteq Q_0 \cap C \subseteq Q_1 \cap C = P_1 \), contradicting INC in \( C \subseteq B \). But \( Q_1 \) minimal over \( q_1 \cap B \) implies that \( Q_1 \) consists of elements which are zero-divisors mod \( q_1 \cap B \) by Lemma 3. So there exists \( x \in B - q_1 \cap B \) so that \( xt \in q_1 \cap B \). This contradicts the fact that \( t \) is regular mod \( q_1 \) in \( R \). Therefore \( I \subset q_1 \), and \( C \subset R \) has GD.

**Corollary (Schelter).** If \( R \) is a prime PI ring integral over an integrally-closed central subring \( A \), then \( A \subset R \) has GD.

**Proof.** By Theorem 2, \( A \subset R \) has GD if and only if \( A \subset R \) has SGD. If \( t \in R \), then \( A[t] \) is a commutative subring of \( R \) with no zero-divisors in \( A \). The commutative going down theorem (see e.g. [6]) may be applied to see that \( A \subset A[t] \) has GD.

**References**

1. Stephen McAdam, Private communication.

Department of Mathematics, University of Southern California, Los Angeles, California 90007