AN APPLICATION OF A THEOREM OF R. E. ZINK

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Abstract. In §1 we discuss a measure theoretic analogue of Blumberg’s theorem; in §2 we discuss a topological analogue of the Saks-Sierpinski theorem.

1. In this section we discuss a measure theoretic analogue of Blumberg’s theorem [7, 1.2]. Suppose $(X, S, \mu)$ is a totally finite measure space [2, p. 73], and $\mu^*$ is the outer measure engendered by $\mu$. We consider the following statement.

1.1. For every real valued function $f$ defined on $X$, there is a subset $D$ of $X$ such that $\mu^*(D) = \mu(X)$ and $f|D$ is measurable ($S \cap D$), where $S \cap D = \{S \cap D : S \in S\}$.

Actually 1.1 is equivalent to a special case of Blumberg’s theorem. Let $(X, S_c, \mu_c)$ denote the completion of $(X, S, \mu)$. By [3, p. 88], there is a topology $\mathbb{T}(\mu_c)$ on $X$ such that (a) if $U \in \mathbb{T}(\mu_c)$ and $U \neq \emptyset$, then $U \in S_c$ and $\mu_c(U) > 0$; and (b) if $A \in S_c$, then there is $U$ in $\mathbb{T}(\mu_c)$ such that $U \subset A$ and $\mu_c(A - U) = 0$. It is easily verified that 1.1 holds for $(X, S, \mu)$ if and only if Blumberg’s theorem holds for $(X, \mathbb{T}(\mu_c))$.

In [7, 2.1] it is shown that

1.2. every subset of the closed unit interval $I$ of cardinality $< 2^{\aleph_0}$ has Lebesgue measure zero,

then 1.1 is false for $(I, S, m)$, where $m$ denotes Lebesgue measure on the collection $S$ of Borel subsets of $I$. In this section we show that the following statement is a consequence of [9, Theorem 9].

1.3. Theorem. Suppose 1.1 is false for $(I, S, m)$. Then 1.1 is false for every separable, nonatomic measure space $(X, S, \mu)$.

Remark. It follows that, in this case, there is $f: X \to R$ such that, if $D$ is a subset of $X$ for which $f|D$ is measurable ($S \cap D$), then $\mu^*(D) = 0$.

If $2^{\aleph_0} = \aleph_1$, then clearly 1.2 holds; therefore, in this case, 1.1 is false for every separable, nonatomic measure space. However, there is a weaker statement—called Martin’s axiom—which implies that 1.2 is true [5, §4].

Presented to the Society May 17, 1976; received by the editors May 29, 1976.


Key words and phrases. Totally finite nonatomic separable measure space, Saks-Sierpinski theorem.

During the period in which this research was done, the author held a Visiting Postdoctoral Fellowship from the Institute for Medicine and Mathematics.

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Martin's axiom. If $C$ is any collection of fewer than $2^{\aleph_0}$ dense open subsets of a compact Hausdorff space which satisfies the countable chain condition, then $\bigcap C \neq \emptyset$.

1.4. Corollary. Suppose Martin's axiom holds. If $(X, \mathcal{F})$ is a quasi-regular [4, p. 157] Baire space which admits a separable, nonatomic category measure [4, p. 156] $\mu$, then Blumberg's theorem does not hold for $(X, \mathcal{F})$.

This corollary follows from 1.3 because Blumberg's theorem holds for $(X, \mathcal{F})$ if and only if 1.1 holds for $(X, \mathcal{S}, \mu)$, where $\mathcal{S}$ denotes the collection of all subsets of $(X, \mathcal{G})$ having the property of Baire.

1.5. Examples. (1) Suppose $\mathcal{G}$ denotes the density topology on the real line $\mathbb{R}$ [7, 2.1]. Then the Stone-Cech compactification of $(\mathbb{R}, \mathcal{G})$ satisfies the hypothesis of 1.4 because $(\mathbb{R}, \mathcal{G})$ does [3, p. 91].

(2) Suppose $(S, \mathcal{U})$ denotes the Stone space of $\mathbb{P}$, where $\mathcal{L}$ is the collection of all Lebesgue measurable subsets of $\mathbb{I}$ and $\mathcal{R} = \{A \in \mathcal{L}: m(A) = 0\}$. Then $(S, \mathcal{U})$ satisfies the hypothesis of 1.4 [3, p. 91].

Remark. It is proven in [6] that if $2^{\aleph_0} = \mathfrak{t}$, then Blumberg's theorem does not hold for $(S, \mathcal{U})$. The preceding provides a measure theoretic proof of this statement.

Proof of 1.3. We shall show that 1.3 follows easily from the next statement, which is a result from [9].

1.6. Theorem (R. E. Zink). Suppose $(X, \mathcal{S}, \mu)$ is a separable, nonatomic, totally finite measure space such that $\mu(X) = 1$. Then there is a function $T: X \to I$ such that (a) $T^{-1}[\mathbb{S}] \subset \mathcal{S}$; (b) if $B \in \mathbb{S}$, then $\mu(T^{-1}[B]) = m(B)$; and (c) if $S \in \mathcal{S}$, then there is $B$ in $\mathbb{S}$ such that $\mu(S \Delta T^{-1}[B]) = 0$, where "$\Delta$" denotes symmetric difference.

To prove 1.3, we assume that 1.1 holds for some separable, nonatomic, totally finite measure space $(X, \mathcal{S}, \mu)$. We assume $\mu(X) = 1$, and that $T$ is as described in 1.6. Suppose $f_0: I \to \mathbb{R}$, and let $f = f_0 \circ T$. Then there is a subset $D_1$ of $X$ such that $\mu^*(D_1) = 1$ and $f|D_1$ is measurable $(\mathbb{S} \cap D_1)$. Because 1.6 (c) holds, there is a subset of $D$ of $D_1$ such that $\mu^*(D) = 1$ and $f|D$ is measurable $T^{-1}[\mathbb{S}] \cap D$. Then $f|T^{-1}[T[D]]$ is measurable $(T^{-1}[\mathbb{S}] \cap T^{-1}[T[D]])$, and $f_0[T[D]$ is measurable $\mathcal{S} \cap T[D]$. Because 1.6 (b) holds, $m^*(T[D]) > \mu^*(D) = 1$; hence 1.1 holds for $(I, \mathbb{S}, m)$.

Questions. (1) Is it consistent with ZF + AC that 1.1 holds for $(I, \mathbb{S}, m)$?

(2) Does 1.3 remain true if the word "separable" is deleted from the hypothesis?

2. In [9], it is shown that the following statement, known as the Saks-Sierpinski theorem, holds for every totally finite measure space $(X, \mathcal{S}, \mu)$.

2.1. For every real valued function $f$ defined on $X$, there is a function $g: X \to \mathbb{R}$ which is measurable $(\mathbb{S})$ such that, for every positive number $\varepsilon$,

$$\mu^*(\{x: |f(x) - g(x)| < \varepsilon\}) = \mu(X).$$
A topological analogue of 2.1 provides another characterization of Baire spaces.

2.2. THEOREM. The following statements are equivalent for any topological space \((X, \mathcal{T})\).

(a) \((X, \mathcal{T})\) is a Baire space.

(b) For every real valued function \(f\) defined on \(X\), there is a function \(g\) such that: (i) domain \(g\) is a dense \(G_δ\) subset of \(X\); (ii) \(g\) is continuous; and (iii) for every positive number \(ε\), the set \(\{x \in \text{domain } g: |f(x) - g(x)| < ε\}\) is dense in \(X\).

(c) For every real valued function \(f\) defined on \(X\), there is a function \(h\) defined on \(X\) such that: (i) \(h\) is Borel measurable; and (ii) for every positive number \(ε\), there is a dense subset \(D_ε\) of \(X\) such that \(h|D_ε\) is continuous and \(D_ε \subseteq \{x: |f(x) - h(x)| < ε\}\).

Proof. Obviously (b) implies (c). If [7, 1.5] is modified by replacing \(Y\) by \(R\) in (3), then the resulting statement is true, and its proof is very similar to the proof of [7, 1.5]. It follows from this modified statement that (c) implies (a). Finally, the proof of the Saks-Sierpinski theorem, which is given in [9, §4], when translated into topological terms, establishes that (a) implies (b), provided the following statement is substituted for Lemma B of [9].

LEMMA B'. Suppose \(f\) is a real valued function defined on the Baire space \(X\), and \(Y\) is a dense subspace of \(X\) such that \(Y\) is a Baire space. For every positive number \(ε\), there is a function \(f_ε\) such that: (i) domain \(f_ε\) is a dense open subset of \(X\); (ii) \(f_ε\) is continuous; and (iii) if \(X_ε = \{x \in \text{domain } f_ε: |f(x) - f_ε(x)| < ε\}\), then \(X_ε \cap Y\) is dense in \(X\) and \(X_ε \cap Y\) is a Baire space.

The proof of Lemma B' is quite similar to the proof that (1) implies (3) of [7, 1.5], and is omitted.

REMARKS. (1) It is clear from the proof of 2.2 that:

(a) The function \(g\) (resp. \(h\)) can be chosen so that for every positive number \(ε\), the set \(\{x \in \text{domain } g: |f(x) - g(x)| < ε\}\) (resp. \(D_ε\)) is a dense Baire subspace of \(X\).

(b) If \(f\) is bounded, then function \(g\) (resp. \(h\)) can be chosen to be bounded.

(c) If \(X\) is completely regular and satisfies the countable chain condition, then \(h\) can be chosen to be a Baire function.

(2) The Saks-Sierpinski theorem, when \(μ\) is complete, is a special case of 2.2 (apply 2.2(c) to the Baire space \((X, \mathcal{T} (μ))\)).

(3) Suppose \((X, \mathcal{T})\) is an extremally disconnected [1, p. 22] Baire space. It follows from 2.2(b) and [1, p. 96] that for every bounded, real valued function \(f\) defined on \(X\), there is a continuous function \(g: X \to R\) such that, for every positive number \(ε\), the set \(\{x: |f(x) - g(x)| < ε\}\) is dense in \(X\). In particular, the preceding statement holds for the space \((S, \mathcal{T})\) of 1.5(2). However, if \(2^ω = ω_1\), then there is a bounded real valued function \(f_0\) defined on \(S\) such
that \( \{ x : f_0(x) = g(x) \} \) is nowhere dense in \( S \) for every continuous real valued function \( g \) defined on \( S \).

**References**


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