A NOTE ON PSEUDOCOMPACT SPACES AND $k_R$-SPACES

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Abstract. Utilizing the Stone-Cech compactification of an uncountable discrete space, we construct a pseudocompact space $X$ which belongs to Frolik's class $\mathbb{P}^*$ but $k_R X$ is not pseudocompact.

All spaces in this paper will be completely regular Hausdorff. Recall that a space $X$ is called a $k_R$-space provided each real-valued function on $X$ is continuous if its restriction to every compact subset of $X$ is continuous, and that associated with each space $X$ there is a unique $k_R$-space $k_R X$ having the same underlying set and the same compact sets as $X$.

Let $\mathcal{R}$ be the class of spaces $X$ such that $k_R X$ is pseudocompact. Let $\mathbb{P}^*$ be the class of pseudocompact spaces $X$ with the property: Each infinite collection of disjoint open sets has an infinite subcollection, each of which meets some fixed compact set $\mathbb{Q}$.

N. Noble [3] showed that $\mathcal{R} \subset \mathbb{P}^*$ and that $\mathbb{P}^*$ is closed under arbitrary products. In [4] he proved $\mathcal{R}$ is also closed under arbitrary products. It was not known, however, whether the two classes coincide or not. The purpose of this note is to show that they differ, i.e., $\mathcal{R}$ is properly contained in $\mathbb{P}^*$.

For a space $X$, $\beta X$ and $X^*$ denote the Stone-Cech compactification of $X$ and its remainder, respectively. For $D$ the discrete set of power $\aleph_1$, let $\Omega$ be the subspace of $D^* = \beta D \setminus D$ consisting of all the elements in the closure (in $\beta D$) of some countable subset of $D$. Let $A$ be a countably infinite, discrete subset of $D^* \setminus \Omega$. Put $X = \Omega \cup A \subset D^*$. Henceforth $X$ denotes this subspace of $D^*$. We will show that $X \in \mathbb{P}^* \setminus \mathcal{R}$.

Assertion 1. $X$ belongs to $\mathbb{P}^*$.

Proof. Since every countable subset of $\Omega$ has compact closure in $\Omega$, $\Omega$ belongs to $\mathbb{P}^*$. Since $\Omega$ is dense in $X$, $X$ also belongs to $\mathbb{P}^*$.

The next property of $D^*$ is the key to prove that $X$ is not in $\mathcal{R}$.

Lemma 1. Let $F$ be a noncompact closed subset of $\Omega$. Then $\text{cl}_{D^*} F \setminus \Omega$ is infinite. In fact, its cardinal is at least $\exp \exp \aleph_1$.

Proof. Let $F$ be a noncompact closed set in $\Omega$. Put $uD = D^* \setminus \Omega$. It is well known that the cardinal of $uD$ is $\exp \exp \aleph_1$ (cf. [5, Theorem 5.13]). Hence,
we need only show that $\text{cl}_D F \setminus \Omega$ contains a copy of $uD$.

Let $x_0 \in F$. Then $x_0 \in \text{cl}_D N_0$ for some countable set $N_0$ in $D$. Let us denote by $\omega_1$ the first uncountable ordinal. Suppose $\alpha < \omega_1$ and that we have chosen a subset $\{x_\gamma\}_{\gamma < \alpha}$ of $F$ and a family $\{N_\gamma\}_{\gamma < \alpha}$ of disjoint countable subsets of $D$ with $x_\gamma \in \text{cl}_D N_\gamma$. Since $\bigcup_{\gamma < \alpha} N_\gamma$ is countable, its closure in $\beta D$ is a compact set contained in $\Omega$; hence $F \setminus \text{cl}_D \bigcup_{\gamma < \alpha} N_\gamma$ is nonempty because $F$ is not compact. Pick a point $x$ from the nonempty set. Since every point of $\Omega$ has a neighborhood (in $\beta D$) which is a closure of some countable subset of $D$, there exists a countable set $N$ in $D$ with $x \in \text{cl}_D N$. Put $x_\alpha = x$ and $N_\alpha = N$.

Thus, by induction, we get a subset $\{x_\alpha\}_{\alpha < \omega_1}$ of $F$ and a family $\{N_\alpha\}_{\alpha < \omega_1}$ of disjoint countable subsets of $D$ such that $x_\alpha \in \text{cl}_D N_\alpha$. Put $F_1 = \{x_\alpha\}_{\alpha < \omega_1}$. Clearly $F_1$ is a copy of $D$. We will show next that $F_1$ is $C^*$-embedded in $\beta D$.

Let $f$ be a bounded real-valued function on $F_1$. Define a function $f_D$ on $D$ by $f_D(N_\alpha) = f(x_\alpha)$ and $f_D(D \setminus \bigcup_{\alpha < \omega_1} N_\alpha) = 0$. Then the Stone extension of $f_D$ over $\beta D$ is an extension of $f$. Hence $F_1$ is $C^*$-embedded in $\beta D$ and this implies $\beta F_1 = \text{cl}_D F_1 \subset \text{cl}_D F_0$. Thus we have $\beta F_1 \setminus \Omega \subset \text{cl}_D F \setminus \Omega$. Now it is easy to see that $\beta F_1 \setminus \Omega$ is a copy of $uD$. This completes the proof.

**Lemma 2.** Every compact subset $K$ of $X = \Omega \cup A$ is a topological sum $K = K_1 \oplus K_2$ of a compact set $K_1$ in $\Omega$ and a finite set $K_2$ in $A$.

**Proof.** Let $K$ be a compact set in $X$. Put $K_1 = K \cap \Omega$ and $K_2 = K \cap A$. Note that $A$ is a closed subset of $X$ because $\Omega$ is locally compact. Therefore $K_2$ is a compact set in the discrete space $A$, i.e., $K_2$ is finite. Since $K$ is compact, $\text{cl}_D K_1 \setminus \Omega$ is contained in the finite set $K_2$. Hence, by Lemma 1, $K_1$ is compact.

**Assertion 2.** $X$ does not belong to $\mathfrak{X}$, i.e., $kRX$ is not pseudocompact.

**Proof.** Let $a \in A$ and let $f_a$ be a real-valued function on $X$ such that $f_a(a) = 1$ and $f_a(x) = 0$ for any $x \neq a$. Then, by Lemma 2, $f_a$ is continuous on every compact subset of $X$. This implies that each point of $A$ is isolated in $kRX$. Since $\Omega$ is locally compact, we have $kRX = \Omega \oplus A$. Now it is clear that $kRX$ is not pseudocompact.

**References**