PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 63, Number 1, March 1977

CONVERGENT SEQUENCES OF τ-SMOOTH MEASURES

SURJIT SINGH KHURANA

ABSTRACT. It is proved that every τ-smooth, group-valued Borel measure on a regular Hausdorff space is regular; also it is proved that if a sequence of τ-smooth Borel measures on a regular Hausdorff space is convergent for regular open sets, then it is convergent for all Borel sets. For a completely regular Hausdorff space, it is proved that if a sequence of Borel τ-smooth measures is convergent for exactly open sets then it is convergent for all Borel sets.

In this paper X denotes a regular Hausdorff topological space, $\mathfrak{B} = \mathfrak{B}(X)$ all Borel subsets of X, G an Hausdorff Abelian complete topological group, and $\mu: \mathfrak{B} \to G$ a countably additive measure. We say that μ is τ-smooth if for any increasing net $\{V_\alpha\}$ of open subsets of X, $\mu(\bigcup V_\alpha) = \lim \mu(V_\alpha)$ [1], [10], [11]. We say that μ is regular if for any $B \in \mathfrak{B}$ and a 0-neighborhood (0-nbd) W of G, there exist a closed set $C \subset B$ and an open set $U \supset B$ such that $\mu(P) \in W$ for any $P \in \mathfrak{B}$ with $P \subset U \setminus C$. An open set U, in X, is called regular if $U = (\bar{U})^\circ$ (here ' $^-$ ' and ' $^\circ$ ' denote, respectively, the closure and interior in X [1], [6], [9]. By an exact open set [1] (or a positive set [11])), we mean a set of the form $\{x \in X: f(x) > 0\}$ for some nonnegative real-valued continuous functions on X. In this paper we prove that every τ-smooth countably additive μ: $\mathfrak{B} \to G$ is regular. Also for a sequence $\{\mu_n\}$ of countably additive τ-smooth measures, we prove that if $\{\mu_n(A)\}$ is convergent for any regular open subset of X, then $\{\mu_n(B)\}$ is convergent for any $B \in \mathfrak{B}$; if X is completely regular we prove that if $\{\mu_n(A)\}$ is convergent for any exact open set A then $\{\mu_N(A)\}$ is convergent for any $B \in \mathfrak{B}$ – in both cases the limits are proved to be countably additive, τ-smooth measures. This generalizes results of [1] (see also [7]).

It should be noted that for any open set U, $(U)^\circ$ is a regular open set [9] and so the class of regular open sets is quite rich. For a family $\mathfrak{M}$ of G-valued measures on $\mathfrak{B}$, a class $\mathcal{C} \subset \mathfrak{B}$ is called a convergence determining class for $\mathfrak{M}$ if every sequence $\{\mu_n\} \subset \mathfrak{M}$, which is pointwise convergent on $\mathcal{C}$, is pointwise convergent on $\mathfrak{B}$. In this terminology we prove that for τ-smooth measures on X, regular open sets is a convergence determining class [9]. For a compact Hausdorff space X every regular Borel G-valued measure

Received by the editors June 11, 1976 and, in revised form, August 13, 1976.

AMS (MOS) subject classifications (1970). Primary 60B05, 46G10, 28A25; Secondary 60B10, 28A45.

Key words and phrases. τ-smooth measures, regular open sets, submeasures, exhaustive submeasures, finitely additive exhaustive measures.

is τ-smooth [7]; in [11] many examples of τ-smooth measures are given.

We shall make use of the theory of submeasures developed in [3]. A mapping $\| \ \|: G \to [0, \infty)$ is called a norm on G if (i) $\|x + y\| \leqslant \|x\| + \|y\|$, (ii) $\|x\| = \|-x\|$, (iii) $\|x\| = 0$ iff $x = 0$, for every x and y in G [3, p. 270]. A group with topology generated by a norm will be called a normed group. It is well known that every complete Abelian Hausdorff topological group is a closed subgroup of a product of complete normed Abelian groups. (To prove this let V be a symmetric 0-neighborhood in G. By induction there exists a sequence $\{U_n\}$ of symmetric 0-neighborhoods such that $U_{n+1} + U_{n+1} + U_{n+1} \subset U_n$ and $U_1 = V$. Proceeding as in [12, Chapter IX, §3.1, Proposition 2] we get a continuous pseudo-norm p_V on G. $G_V = p_V^{-1}\{0\}$ is a closed subgroup of G. It is easy to see that G is embedded in $\Pi\{G/G_V: V$ a symmetric 0-nbd$\}$, considering G/G_V a normed group with norm coming from the pseudo-norm p_V.)

For a σ-algebra $\mathfrak{A}$, of subsets of a set Y, a finitely additive measure λ: $\mathfrak{A} \to G$ is called exhaustive [3] if for any disjoint sequence $\{B_i\} \subset \mathfrak{A}$, $\lambda(B_i) \to 0$, in G. If λ is countably additive then it is evidently exhaustive. A class λ_α: $\mathfrak{A} \to G$ is called uniformly exhaustive if for any disjoint sequence $\{A_i\} \subset \mathfrak{A}$, $\lambda_\alpha(A_i) \to 0$, uniformly in α. If $(H, \| \cdot \|)$ is a normed group and λ: $\mathfrak{A} \to H$ is an exhaustive finitely additive measure, then we get an exhaustive submeasure $\bar{\lambda}$: $\mathfrak{A} \to [0, \infty)$, $\bar{\lambda}(B) = \sup\{\|\lambda(A)\|: A \subset B, A \in \mathfrak{A}\}$, $\forall B \in \mathfrak{A}$ (it is proved in [3, Corollary 4.11, p. 279], that $\bar{\lambda}$ is finite valued).

We first present the following lemmas:

LEMMA 1. *Let λ_n: $2^N \to G$ be a sequence of countably additive measures such that $\{\lambda_n(M)\}$ converges, $\forall M \subset N$. Then $\{\lambda_n\}$ is uniformly convergent on 2^N.*

PROOF. This lemma is proved in [7] (cf. [4]).

LEMMA 2. *Let $\lambda_n: 2^N \to G$ be a sequence of finitely additive, exhaustive measures. If $\{\lambda_n(M)\}$ is convergent, $\forall M \subset N$, then λ_n's are uniformly exhaustive.*

PROOF. This is proved in [4, p. 726].

LEMMA 3. *Let $\mu: \mathfrak{B} \to G$ be countably additive and τ-smooth and $(G, \| \ \|)$ be normed. Then for any open set V, in X, there exists an increasing sequence $\{C_n\}$ of closed sets, $C_n \subset V$, $\forall n$, such that $\mu \equiv 0$ on $(V \setminus \bigcap_{n=1}^{\infty} C_n) \cap \mathfrak{B}$. Dually for any closed set C, in X, there is a decreasing sequence $\{V_n\}$ of open sets, such that $V_n \supset C$, $\forall n$ and $\mu \equiv 0$ on $(\bigcap_{n=1}^{\infty} V_n \setminus C) \cap \mathfrak{B}$.*

PROOF. We split the proof into several steps.

(i) For an open set U and $\varepsilon < 0$, there exists a closed set $C \subset U$ such that $\|\mu(U \setminus C)\| \leqslant \varepsilon$. Suppose this is not true. Since μ is τ-smooth there exists an open set $U_1 \subset U$ such that $\bar{U}_1 \subset U$, $\|\mu(U \setminus U_1)\| < \varepsilon/8$ (we are using that X is regular), and $\|\mu(U \setminus \bar{U}_1)\| > \varepsilon$. This implies that $\|\mu(\bar{U}_1 \setminus U_1)\| > \varepsilon/2$.

Similarly we get an open set $U_2 \subset U_1$, $\overline{U}_2 \subset U_1$ such that $\|\mu(U_1 \setminus U_2)\| < \varepsilon/8^2$. This means

$$\left|\mu\left(\overline{U}_2 \setminus U_2\right)\right| > \left\|\mu\left(U \setminus \overline{U}_2\right)\right\| - \|\mu(U \setminus U_1)\| + \|\mu(U_1 \setminus U_2)\|$$
$$> \varepsilon - \varepsilon/2 - \varepsilon/2.$$

In this fashion we get a sequence $\{U_n\}$ of open sets such that $\overline{U}_{n+1} \subset U_n$ and $\|\mu(\overline{U}_n \setminus U_n)\| > \varepsilon/2$. Since $\{\overline{U}_n \setminus U_n\}$ are mutually disjoint, this is a contradiction.

(ii) For an open set U and $\varepsilon > 0$, there exists a closed set $C \subset U$ such that $\sup\{\|\mu(P)\|: P$ closed, $P \subset U \setminus C\} \leqslant \varepsilon$. An indirect proof may be obtained by routine arguments.

REMARK. If the above result holds for a closed set C, then it also holds for a bigger closed set.

(iii) If for any open set U and $\varepsilon > 0$, $\sup\{\|\mu(P)\|: P$ closed, $P \subset U\} \leqslant \varepsilon$, then $\sup\{\|\mu(V)\|: V$ open, $V \subset U\} \leqslant \varepsilon$.

PROOF. Use (i).

(iv) For any open set V there exists an increasing sequence $\{C_n\}$ of closed sets, $C_n \subset V$, $\forall n$ such that $\mu \equiv 0$ on $(V \setminus \bigcup_{n=1}^{\infty} C_n) \cap \mathfrak{B}$.

PROOF. By (ii), there exists an increasing sequence $\{C_n\}$ of closed sets, $C_n \subset V$, $\forall n$ such that $\sup\{\|\mu(P)\|: P$ closed, $P \subset V \setminus C_n\} \leqslant 1/n$, $\forall n$. By (iii), for any open set U, $\|\mu(U \cap (V \setminus C_n))\| \leqslant 1/n$, $\forall n$ and so

$$\mu\left[U \cap \left(V \setminus \bigcup_{n=1}^{\infty} C_n\right)\right] = 0.$$

Since $\mathfrak{A}_0 = \{U \cap (V \setminus \bigcup_{n=1}^{\infty} C_n): U$ open in $X\}$ is a lattice, by [8, p. 188], $\mu \equiv 0$ on the algebra generated by $\mathfrak{A}_0$ and so $\mu \equiv 0$ on the σ-algebra generated by $\mathfrak{A}_0$. This proves $\mu \equiv 0$ on $(V \setminus \bigcup_{n=1}^{\infty} C_n) \cap \mathfrak{B}$. The dual statement is proved by routine arguments.

LEMMA 4. *Let μ: $\mathfrak{B} \to G$ be a countably additive τ-smooth measure, G being normed. Then:*

(i) *$\bar{\mu}$ is order continuous (i.e., for any sequence $\{B_i\} \subset \mathfrak{B}$, with $B_i \downarrow \varnothing$, $\bar{\mu}(B_i) \downarrow 0$), exhaustive, σ-subadditive, and finite-valued submeasure* ([3, *Corollary* 4.11, *p.* 279], *also* §5).

(ii) *$\bar{\mu}$ is regular, i.e., for any $B \in \mathfrak{B}$ and $\varepsilon > 0$, there exists a closed set C and an open U such that $C \subset B \subset U$ and $\bar{\mu}(U \setminus C) \leqslant \varepsilon$.*

(iii) *For a closed set $C \subset X$ and $\varepsilon > 0$, there exists a regular open set $U \supset C$ such that $\bar{\mu}(U \setminus C) \leqslant \varepsilon$.*

PROOF. (i) Proof of this part is straightforward (see [3]).

(ii) Take any open set V and fix $\varepsilon > 0$. By Lemma 3 there exists an increasing sequence $\{C_n\}$ of closed sets such that on $(V \setminus \bigcup_{n=1}^{\infty} C_n) \cap \mathfrak{B}$, $\mu \equiv 0$. Thus $\bar{\mu}(V \setminus V_0) = 0$, where $V_0 = \bigcup_{n=1}^{\infty} C_n$. Since $\bar{\mu}$ is order continuous there exists an $n_0 \in N$ such that $\mu(V_0 \setminus C_{n_0}) < \varepsilon$. Now

$$\bar{\mu}(V \setminus C_{n_0}) \leqslant \bar{\mu}(V \setminus V_0) + \bar{\mu}(V_0 \setminus C_{n_0}) < \varepsilon.$$

Similarly for a closed set C, there exists an open set $U \supset C$ such that $\bar{\mu}(U \setminus C) < \varepsilon$. Let $\mathcal{C} = \{A \in \mathfrak{B} : \text{Given } \varepsilon > 0, \exists$ a closed set C and an open set U, $C \subset A \subset U$, with $\bar{\mu}(U \setminus C) < \varepsilon\}$. Proceeding exactly as in [2, p. 93], we see that $\mathcal{C}$ is a σ-ring, containing all closed sets and all open sets. Thus $\mathcal{C} = \mathfrak{B}$.

(iii) Suppose this is not true. Take a $V_1 \in \mathfrak{U}$, the class of all regular open sets in X, $V_1 \supset C$. Since $\bar{\mu}(V_1 \setminus C) > \varepsilon$, by (ii) there exists an open set $W_1 \subset V_1 \setminus C$ such that $\| \mu(W_1)\| > \varepsilon$. Using (ii) and τ-smoothness of μ, there exists a $U_1 \in \mathfrak{U}$ such that $\overline{U}_1 \subset W_1$ and $|\mu(U_1)| > \varepsilon$. Since $\mathfrak{U}$ is closed under finite intersection, $V_1 \cap (X \setminus \overline{U}_1) \in \mathfrak{U}$ [5, p. 92, Problem 22]. Proceeding as above we get a disjoint sequence $\{U_i\} \subset \mathfrak{U}$, such that $|\mu(U_i)| > \varepsilon$, $\forall i$. This is a contradiction.

THEOREM 5. *Every countably additive τ-smooth measure μ: $\mathfrak{B} \to G$ is regular.*

PROOF. Since G is a subgroup of a product of normed groups the result follows from Lemma 4.

THEOREM 6. *Let μ_n: $\mathfrak{B} \to G$ be a sequence of countable additive τ-smooth measures such that* lim $\mu_n(A)$ *exists for any regular open set A. Then*

$$\lim \mu_n(B) = \mu(B)$$

exists for any $B \in \mathfrak{B}$ and μ is also countably additive τ-smooth.

PROOF. The result will be proved if proved under the assumption that G is normed. Let $\{U_i\}$ be a disjoint sequence in $\mathfrak{U}$, the class of regular open sets in X; then λ_n: $2^N \to G$, $\lambda_n(M) = \mu_n(\overline{\bigcup_{i \in M} U_i}^{\circ})$ is a sequence of exhaustively finitely additive measures (exhaustivity is the consequence of the fact that for any disjoint sequence $\{M_n\}$ in 2^N, $\{\overline{\bigcup_{i \in M_n} U_i}^{\circ}\}$ is a disjoint sequence) such that $\lim \lambda_n(M)$ exists for every $M \subset N$. By Lemma 2, λ_n's are uniformly exhaustive. We claim that given $\varepsilon > 0$ and a closed set C there exists a $U \in \mathfrak{U}$ such that $\bar{\mu}_n(U \setminus C) \leqslant \varepsilon$, $\forall n$. Suppose this is not true. Take a $V_1 \in \mathfrak{U}$, $V_1 \supset C$, such that $\bar{\mu}_1(V_1 \setminus C) \leqslant \varepsilon$. This means, using regularity, there exists an open set $W_1 \subset V_1 \setminus C_1$ and an $n(1) \in N$, $n(1) > 1$, such that $\| \mu_{n(1)}(W_1)\| > \varepsilon$. Since X is regular and $\mu_{n(1)}$ is τ-smooth, there exists a $U_1 \in \mathfrak{U}$, $\overline{U}_1 \subset W_1$ such that $\| \mu_{n(1)}(U_1)\| > \varepsilon$. Since $V_1 \cap (X \setminus \overline{U}_1) \in \mathfrak{U}$ [5, p. 92, Problem 22], proceeding as above we get a disjoint sequence $\{U_i\} \subset \mathfrak{U}$ and a strictly increasing sequence $\{n(i)\} \subset N$, such that $\| \mu_{n(i)}(U_i)\| > \varepsilon$, $\forall i$. This is a contradiction since in the above terminology $\lambda_{n(i)}$'s are uniformly exhaustive.

For any open set U and $\varepsilon > 0$, $\exists$ a closed set $C \subset U$ such that $\bar{\mu}_n(U \setminus C) \leqslant \varepsilon$, $\forall n$, for otherwise, proceeding as above, we get a disjoint sequence $\{U_i\} \subset \mathfrak{U}$ and a strictly increasing sequence $\{n(i)\} \subset N$, such that

$$\overline{U}_{i+1} \subset X \setminus \overline{U}_i \quad \text{and} \quad \| \mu_{n(i)}(U_i)\| > \varepsilon, \quad \forall i,$$

a contradiction. Using these facts we see that $\{\mu_n(U)\}$ is convergent for any open set U. For an $\varepsilon > 0$ and a disjoint sequence $\{C_i\}$ of closed sets, take a sequence $\{U_i\}$ of open sets such that $\bar{\mu}_n(U_i \setminus C_i) \leqslant \varepsilon/2^{i+2}$, for every i and for every n. From

$$\left\|(\mu_m - \mu_n)\left(\bigcup_{i\in M} C_i\right)\right\| \leqslant \left\|(\mu_m - \mu_n)\left(\bigcup_{i\in M} U_i\right)\right\| + \bar{\mu}_m\left(\bigcup_{i\in M}(U_i \setminus C_i)\right) + \bar{\mu}_n\left(\bigcup_{i\in M}(U_i \setminus C_i)\right) \leqslant \left\|(\mu_m - \mu_n)\left(\bigcup_{i\in M} U_i\right)\right\| + \frac{\varepsilon}{2},$$

it follows that $\mu_n(\bigcup_{i\in M} C_i)$ is convergent $\forall M \subset N$, and so by Lemma 1, $\mu_n(C_i) \to 0$, uniformly in n. Using this result and proceeding in a similar way we prove that for $\varepsilon > 0$ and a $B \in \mathfrak{B}$, $\exists$ a closed set $C \subset B$ such that $\bar{\mu}_n(B \setminus C) \leqslant \varepsilon$. This proves $\{\mu_n(B)\}$ is convergent for any $B \in \mathfrak{B}$ (see [7], [9]).

The countable additivity of the limit measure μ follows from Lemma 1. We will prove that μ_n's are uniformly τ-smooth; this will prove that the limit measure is τ-smooth. Let $\{V_\alpha\}_{\alpha\in I}$ be an increasing net of open sets with $V = \bigcup V_\alpha$. If $\{\mu_n\}$ are not uniformly τ-smooth, by taking subsequences if necessary there exists an increasing sequence $\alpha(n) \subset I$, such that

$$\|\mu_n(V_{\alpha(n+1)} \setminus V_{\alpha(n)})\| > \varepsilon,$$

for every n, for some $\varepsilon > 0$. Since $\{V_{\alpha(n+1)} \setminus V_{\alpha(n)}\}$ are mutually disjoint and $\{\mu_n(B)\}$ is convergent, $\forall B \in \mathfrak{B}$, this is a contradiction, by Lemma 1.

THEOREM 7. *Let X be a completely regular Hausdorff space and $\mu_n\colon \mathfrak{B} \to G$ a sequence of τ-smooth, countably additive measure. If $\{\mu_n(U)\}$ is convergent for every exactly open set U in X, then $\{\mu_n(B)\}$ is convergent for every $B \in \mathfrak{B}$.*

PROOF. As before, we can assume G to be normed. Let $\tilde{X}$ be the Stone-Čech compactification of X and $\tilde{\mathfrak{B}}$ all Borel subsets of $\tilde{X}$. $\tilde{\mu}\colon \tilde{\mathfrak{B}} \to G$, $\tilde{\mu}_n(B) = \mu_n(B \cap X)$, $B \in \tilde{\mathfrak{B}}$, are countably additive and τ-smooth. By Lemma 4, $\{\bar{\tilde{\mu}}_n\}$ are regular. Thus given m and n, in N, and an open set V, in $\tilde{X}$, there exists a compact set $C_{m,n} \subset V$ such that $\bar{\tilde{\mu}}_n(V \setminus C_{m,n}) < 1/m$. Take any exact open set $U_{m,n}$, $C_{m,n} \subset U_{m,n} \subset V$. This means $\bar{\tilde{\mu}}_n(V \setminus U) = 0$, where $U = \bigcup_{m,n} U_{m,n}$. Since $\|\tilde{\mu}_n(V \setminus U)\| \leqslant \bar{\tilde{\mu}}_n(V \setminus U) = 0$, $\forall n$ and U is exactly open we get that $\{\tilde{\mu}_n(V)\}$ is convergent for every open set V in $\tilde{X}$. By Theorem 6, $\{\tilde{\mu}_n(B)\}$ is convergent, $\forall B \in \tilde{\mathfrak{B}}$. From this it follows that $\{\mu_n(B)\}$ is convergent $\forall B \in \mathfrak{B}$.

I am grateful to the referee for making several useful suggestions.

REFERENCES

1. W. Adamski, P. Gänssler and S. Kaiser, *On compactness and convergence in the space of measures*, Math. Ann. **220** (1976), 193–210.

2. N. Dinculeanu and P. W. Lewis, *Regularity of Baire measures*, Proc. Amer. Math. Soc. **26** (1970), 92–94. MR **41** #5588.

3. L. Drewnowski, *Topological rings of sets, continuous set functions, integration*. I, II, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **20** (1972), 269–276; 277–286. MR **46** #5558.

4. ______, *Equivalence of Brooks-Jewett, Vitali-Hahn-Saks and Nikodym theorems*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **20** (1972), 725–731. MR **47** #431.

5. J. Dugundji, *Topology*, Allyn and Bacon, Boston, Mass., 1966. MR **33** #1824.

6. P. R. Halmos, *Lectures on Boolean algebras*, Van Nostrand, Princeton, N.J., 1963. MR **29** #4713.

7. S. S. Khurana, *Convergent sequences of regular measures*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **24** (1976), 37–42.

8. B. J. Pettis, *On the extension of measures*, Ann. of Math. (2) **54** (1951), 186–197. MR **13**, 19.

9. L. Rogge, *The convergence determining class of regular open sets*, Proc. Amer. Math. Soc. **37** (1973), 581–585. MR **47** #434.

10. F. Topsøe, *Topology and measure*, Lecture Notes in Math., vol.133, Springer-Verlag, Berlin and New York, 1970.

11. V. S. Varadarajan, *Measures on topological spaces*, Mat. Sb. **55(97)** (1961), 35–100; English transl., Amer. Math. Soc. Transl. (2) **48** (1965), 161–228. MR **26** #6342.

12. N. Bourbaki, *Elements of mathematics. General Topology*. Part 2, Hermann, Paris; English transl., Addison-Wesley, Reading, Mass., 1966. MR **34** #5044b; erratum, **40**, p. 1704.

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF IOWA, IOWA CITY, IOWA 52242