ON CONTINUITY OF FIXED POINTS OF COLLECTIVELY CONDENSING MAPS

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ABSTRACT. In this paper, we prove, in two parts, the following claim. Let $X$ be a Banach space and $\Lambda$ an arbitrary topological space. Suppose that $T: \Lambda \times X \to X$ is collectively condensing; then the fixed point set $S(\lambda, y)$ has closed graph if and only if $T$ is continuous in both $\lambda$ and $y$.

Zvi Artstein [1] proved

**Theorem 0.** Suppose $X$ is a Banach space and $\Lambda$ a metric space. Let $T: \Lambda \times X \to X$ be collectively condensing, i.e., for every $B \subset X$,

$$\chi\left(\bigcup_{\lambda \in \Lambda} T(\lambda, B)\right) \leq \chi(B),$$

where equality implies $\chi(B) = 0$ and $\chi$ is the Kuratowski measure of noncompactness, defined as $\chi(B) = \inf\{d | B \text{ can be covered by a finite number of subsets of diameter } < d\}$. Let $S(\lambda, y) = \{x \in X | x = T(\lambda, x) + y\}$. Then, $S(\lambda, y)$ is upper-semicontinuous if and only if $T(\lambda, x)$ is continuous in $\lambda$ and $x$ simultaneously.

At the end of his paper, he posed an open question. Is Theorem 0 still true if $\Lambda$ is a general topological space? Upon investigation we find that $S(\lambda, y)$ has closed graph if and only if $T(\lambda, x)$ is continuous in both $\lambda$ and $x$.

The following definitions are intended to refresh the memory of the readers.

**Definition 0.1.** Let $F, \Lambda$ be topological spaces, and $A$ a subset of a topological space $E_2$. We say a multifunction $F: A \to F, \Lambda$ is upper-semicontinuous at $A_0 \in A$ if for each open set $G$ in $E_2$ containing $F(A_0)$, there exists an open neighborhood $U(A_0)$ of $A_0$ in $E_2$ such that $F(U(A_0) \cap A) \subset G$.

**Definition 0.2.** $F$ is upper-semicontinuous on $A$ if it is upper-semicontinuous at each $A_0 \in A$.

To prove our main theorem, we need the following tools.

**Theorem 0.3** [2, Theorem 11.5]. A net has $y$ as a cluster point iff it has a subnet which converges to $y$.

**Theorem 0.4** [2, Theorem 11.8]. Let $f: X \to Y$. Then $f$ is continuous at $x_0 \in X$ iff whenever $x_\lambda \to x_0$ in $X$, then $f(x_\lambda) \to f(x_0)$ in $Y$.

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Theorem 0.5 [2, Theorem 11Dc]. If every subnet of a net \((x_\lambda)\) has a subnet converging to \(x\), then \((x_\lambda)\) converges to \(x\).

The following are our main theorems.

Theorem 1. Let \(X\) be a Banach space, and \(\Lambda\) a topological space. Let \(T: \Lambda \times X \to X\) be collectively condensing. Suppose that \(S\) is upper-semicontinuous; then \(T\) is continuous in \(\Lambda\) and \(x\).

Proof. We will denote \(T(\lambda, \cdot)\) by \(T_\lambda\). Let \((\lambda_\alpha, x_\alpha)\) be a net in \(\Lambda \times X\) converging to \((\lambda_0, x_0)\). We will show that there exists a subnet \((T_{\lambda_\alpha}, x_{\lambda_\alpha})\) converging to \((T_{\lambda_0}, x_0)\). Let \(r_n = 1/2^n, n = 1, 2, \ldots\). Let \(S_0 = \{x_\alpha \mid \|x_\alpha - x_0\| \leq r\}\). We claim \(S_0\) is a subnet of \((x_\alpha)\). Indeed, a routine argument shows that the set \(A_0 = \{\alpha \mid x_\alpha \in S_0\}\) is a directed set, cofinal with the given directed set. Consequently, \(\{(\lambda_\alpha, x_\alpha) \mid \alpha \in A_0\}\) is a subnet, whereas \(S_0\) is a subnet of \((x_\alpha)\) and \((T_{\lambda_0}, x_0)\), respectively.

Let \(x_\alpha, x_\beta \in S_0\). Then \(\|x_\alpha - x_\beta\| \leq \|x_\alpha - x_0\| + \|x_\beta - x_0\| \leq 2r_1\). Hence, \(\text{diam}(S_0) \leq 2r_1\). By collective condensingness, \(\chi((T_{\lambda_\alpha}(x_\alpha) \mid \alpha \in A_0))\) \(\leq \chi(S_0) \leq 2r_1\). Let \(d_1 = \chi((T_{\lambda_\alpha}(x_\alpha) \mid \alpha \in A_0))\). Choose \(\varepsilon_1 > 0\) so that \(d_1 + \varepsilon_1 < 2r_1\). Then, there exists a finite cover: \(S_1, S_2, \ldots, S_k\) of \(S_0 = \{T_{\lambda_\alpha}(x_\alpha) \mid \alpha \in A_0\}\). We claim that there is a set \(S_1, 1 \leq j \leq k(1)\), which contains a subnet of \(S_0\). The following is the proof of this claim. Let \(S = \bigcup_{i=2}^{k(1)} S_i, A_S = \{\alpha \in A_1 \mid T_{\lambda_\alpha}(x_\alpha) \in S\}\), and \(A_{S_j} = \{\alpha \in A_1 \mid T_{\lambda_\alpha}(x_\alpha) \in S_j\}\). Then \(S_1, S\) is a cover of \(S_0\), and \(A_{S_j} \cup A_S = A_1\). Suppose that \(S_1\) does not contain a subnet of \(S_0\). Then there exists \(\alpha \in A_1\) such that \(d \not\geq \alpha\) implies that \(d \not\in A_{S_j}\). Let \(D = \{d \in A_1 \mid d \geq \alpha\}\) for some \(\alpha \in A_1\), and if \(d \not\geq d, d \not\in D\).

We will show that \(D\) is a directed set cofinal with \(A_1\). Clearly, \(D \subseteq A_S\). We need only show that if \(\alpha \in A\), then there exists a \(d \in D\) such that \(d \geq \alpha\).

Since \(A_1\) is a directed set, it follows that there exists a \(\alpha \in A_1\) such that \(d \geq \alpha\).

Let \(D = \{d \in A_1 \mid d \geq \alpha\}\). Clearly, \(D\) is a directed set cofinal with \(A_1\). Since \(S_1\) does not contain a subnet of \(S_0\), we can find \(d^* \in A_1, d^* \not\geq \alpha\) such that \(d \not\geq d^*\) implies \(d \not\in A_{S_j}\). Hence, \(d^* \in D\). Thus, \(D\) is a directed set cofinal with \(A_1\). This means that if \(S_1\) does not contain a subnet, then \(S\) must contain one. By repeating the argument a finite number of times we see that there exists \(S_j, 1 \leq j \leq k(1)\), which contains a subnet of \(S_0\). Let \(S_1 = S_j\). Let \(\mathfrak{K}_1\) denote the subnet contained in \(S_1\). Similarly, we can define \(\mathfrak{K}_2, \mathfrak{K}_3, \ldots\). Clearly, \(\mathfrak{K}_1 \supset \mathfrak{K}_2 \supset \cdots\), where \(\mathfrak{K}_n\) is of diameter \(2r_n = 2/2^n = 1/2^{n-1}, n = 1, 2, \ldots\). Thus \(\bigcap_{n=1}^{\infty} \mathfrak{K}_n = x\) for some \(x \in X\). Let \(B_{r_n}(x)\) be an open ball in \(X\) with center at \(x\) and radius \(r_n\). Then \(B_{r_n}(x)\) contains a subnet of \((T_{\lambda_\alpha}(x_\alpha))\). This can be seen as follows. Consider \(B_{r_{n+1}}(x_\alpha)\). By construction, there exists \(T_{\lambda_\alpha}(x_\alpha) \in \mathfrak{K}_{n+1}\) such that \(T_{\lambda_\alpha}(x_\alpha) \in B_{r_{n+1}}(x)\). Thus, for each \(T_{\lambda_\alpha}(x_\alpha) \in \mathfrak{K}_{n+1}\),

\[\text{diam}(\mathfrak{K}_{n+1}) \leq 2r_{n+1}\]

The proof of this claim is essentially due to Jack Porter.
\[
\|T_{x_\alpha}(x) - x\| \leq \|T_{x_\alpha}(x) - T_{x_{\alpha+2}}(x_{\alpha+2})\| + \|T_{x_{\alpha+2}}(x_{\alpha+2}) - x\| < 2r_{n+1} = r_n,
\]
i.e., \(R_{n+2} \subset B_{r_n}(x)\). This means, in other words, \(x\) is a cluster point of \(R_1\).

Hence, by Theorem 0.3, there is a subnet \(R\) of \(R_1\) which converges to \(x\). Let \(R = (T_{x_{\alpha}}(x))\). Then, clearly, \((\lambda_{\alpha}, x_{\alpha})\) \(\to\) \((\lambda_0, x_0)\). Let \(y = x_0 - x\), and for each \(\alpha\), let \(y_{\alpha} = x_{\alpha} - T_{x_{\alpha}}(x_{\alpha})\). Then, \((y_{\alpha}) \to y\), for some \(y \in X\). By hypothesis, \(x_0 \in S(\lambda_0, y)\), i.e., \(x_0 = T_{\lambda_0}(x_0) - y\). Thus, \(x = T_{\lambda_0}(x_0)\). Hence, \((T_{x_{\alpha}}(x_\alpha)) \to (T_{\lambda_0}(x_0))\). This actually shows that for each subnet of \((T_{x_{\alpha}}, x_\alpha)\), there is a subnet converging to \(T_{\lambda_0}(x_0)\) by Theorem 0.5, \((T_{\lambda_0}(x))\) \(\to\) \(T_{\lambda_0}(x_0)\). By Theorem 0.4, \(T\) is continuous in \((\lambda, x)\). This completes the proof of Theorem 1.

**Remark 1.** The above theorem is still true if \(S\) has closed graph instead. The proof is essentially the same.

**Theorem 2.** Let \(X\) be a Banach space, and \(\Lambda\) a topological space. Let \(T: \Lambda \times X \to X\) be continuous. For each \((\lambda, y) \in \Lambda \times X\), let \(S(\lambda, y) = \{x \in X | x = T(\lambda, x) + y\}\). Then \(S\) has closed graph.

**Proof.** Let \((\lambda_\alpha, y_\alpha)\) be a net converging to \((\lambda_0, y_0)\). For each \(\alpha\) let \(x_\alpha \in S(\lambda_\alpha, y_\alpha)\), i.e., \(x_\alpha = T(\lambda_\alpha, x_\alpha) + y_\alpha\). We will show that if \(\bar{x}\) is a cluster point of \((x_\alpha)\), then \(\bar{x} \in S(\lambda_0, y_0)\). Let \((\lambda_{\alpha})\) be a subnet of \((\lambda_\alpha)\) converging to \(\bar{x}\). Then, clearly, \((\lambda_{\alpha}) \to \lambda_0\), \((y_{\alpha}) \to y_0\). Hence, \((T(\lambda_{\alpha}, x_{\alpha})) = (x_{\alpha} - y_{\alpha}) \to \bar{x} - y_0\). By continuity, \((T(\lambda_{\alpha}, x_{\alpha})) \to T(\lambda_0, \bar{x})\). By Hausdorff property, \(T(\lambda_0, \bar{x}) = \bar{x} - y_0\), i.e., \(\bar{x} = T(\lambda_0, \bar{x}) + y_0\), or \(\bar{x} \in S(\lambda_0, y_0)\). Hence, \(S\) has closed graph.

**Remark 2.** Fred S. Van Vleck proved that if \(X\) is a Euclidean space, then the result of Theorem 0 is still valid no matter what space \(\Lambda\) is. However, if \(X\) is an arbitrary Banach space, our conjecture is that \(S\) need not be upper-continuous even though \(T\) is continuous and collectively condensing.

**References**


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