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FUNCTIONAL EQUATIONS FOR POLYNOMIALS

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ABSTRACT. The set of all continuous symmetric multilinear forms of degree m on a real topological vector space V are shown to be in one-to-one correspondence with the family of continuous scalar-valued functions on V satisfying a certain functional equation. If V is n-dimensional, these functions are precisely those which can be represented by m-homogeneous polynomials of degree n (with respect to some basis of V).

The connection between this family of generalized polynomials and the mth derivatives of a scalar-valued function is discussed.

Denote by N the set of natural numbers and, for each $n \in N$, by $\underline{n}$ the set $\{m \in N\colon m \leqslant n\}$. For all $m, n \in N$, write $\underline{n}^{\underline{m}}\uparrow$ for the family of increasing functions in $\underline{n}^{\underline{m}}$.

Let A be an Abelian group and B a linear space over a field of characteristic 0. The binary operations in A and B will both be denoted $+$ and, for each $n \in N$ and $x \in A \cup B$, nx will signify $x + x + \cdots + x$ (n times). For each $m \in N$, $\mathrm{sm}_m(A, B)$ will be the set of all symmetric functions $\phi|A^{\underline{m}} \to B$ such that, whenever $k \in \underline{m}$ and $x, y, z \in A^{\underline{m}}$ satisfy $x_k + y_k = z_k$ and $x_j = y_j = z_j$ for all $j \neq k$, then $\phi(x) + \phi(y) = \phi(z)$.

For each $m \in N$, define $F_m|B^A \times A^{\underline{m}} \to B$ by letting

$$F_m(f, x) = \sum_{k \in \underline{m}} (-1)^{m-k} \sum_{\sigma \in \underline{m}^{\underline{k}}\uparrow} f\left(\sum_{j \in \underline{k}} x_{\sigma(j)}\right)$$

for all $f \in B^A$ and $x \in A^{\underline{m}}$. Define $l_m(A, B)$ to be the set of all $f \in B^A$ such that $f(kx) = k^m f(x)$ and $F_{m+1}(f, y) = 0$ for all $k \in \underline{n}$, $x \in A$, and $y \in A^{\underline{m+1}}$. Note that $l_1(A, B) = \mathrm{HOM}(A, B) = \mathrm{sm}_1(A, B)$. We shall show that $l_m(A, B)$ and $\mathrm{sm}_m(A, B)$ are isomorphic for larger m as well.

For $m \in N$ and $x \in A^{\underline{m+1}}$, we shall write x', x'', and x''' for the elements of $A^{\underline{m}}$ satisfying $x_j = x'_j = x''_j = x'''_j$ for $j \in \underline{m-1}$ and $x'_m = x_m$, $x''_m = x_{m+1}$, and $x'''_m = x_m + x_{m+1}$.

LEMMA. *For all $m \in N$, $x \in A^{\underline{m+1}}$, and $f \in B^A$,*

$$F_{m+1}(f, x) = F_m(f, x''') - F_m(f, x'') - F_m(f, x').$$

PROOF. Define

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$$\theta_1 \equiv \sum_{k \in \underline{m-1}} (-1)^{m-k} \sum_{\sigma \in \underline{m-1}^{\underline{k}}_{\uparrow}} f\left(\sum_{j \in \underline{k}} x_{\sigma(j)} \right),$$

$$\theta_2 \equiv \sum_{k \in \underline{m-1}} (-1)^{m-k} \sum_{\substack{\sigma \in \underline{m}^{\underline{k}}_{\uparrow} \\ \sigma(k)=m}} f\left(\sum_{j \in \underline{k}} x'_{\sigma(j)} \right),$$

$$\theta_2 \equiv \sum_{k \in \underline{m-1}} (-1)^{m-k} \sum_{\substack{\sigma \in \underline{m}^{\underline{k}}_{\uparrow} \\ \sigma(k)=m}} f\left(\sum_{j \in \underline{k}} x'''_{\sigma}(j) \right),$$

$$\theta_4 \equiv \sum_{k \in \underline{m-1}} (-1)^{m-k} \sum_{\substack{\sigma \in \underline{m}^{\underline{k}}_{\uparrow} \\ \sigma(k)=m}} f\left(\sum_{j \in \underline{k}} x''_{\sigma}(j) \right),$$

$$\theta_5 \equiv f\left(\sum_{j \in \underline{m}} x'''_j \right), \quad \theta_6 \equiv f\left(\sum_{j \in \underline{m}} x''_j \right), \quad \text{and} \quad \theta_7 \equiv f\left(\sum_{j \in \underline{m}} x'_j \right).$$

The following equalities are evident:

$$-\theta_1 = \sum_{k=1}^{m-1} (-1)^{m+1-k} \sum_{\sigma \in \underline{m-1}^{\underline{k}}_{\uparrow}} f\left(\sum_{j \in \underline{k}} x_{\sigma(j)} \right),$$

$$\theta_2 = \sum_{k=2}^{m} (-1)^{m+1-k} \sum_{\substack{\sigma \in \underline{m+1}^{\underline{k}}_{\uparrow} \\ \sigma(k)=m+1 \\ \sigma(k-1)=m}} f\left(\sum_{j \in \underline{k}} x_{\sigma(j)} \right),$$

$$-\theta_3 = \sum_{k=1}^{m-1} (-1)^{m+1-k} \sum_{\substack{\sigma \in \underline{m+1}^{\underline{k}}_{\uparrow} \\ \sigma(k)=m+1 \\ \sigma(k-1)\neq m}} f\left(\sum_{j \in \underline{k}} x_{\sigma(j)} \right),$$

$$-\theta_4 = \sum_{k=1}^{m-1} (-1)^{m+1-k} \sum_{\substack{\sigma \in \underline{m+1}^{\underline{k}}_{\uparrow} \\ \sigma(k)=m}} f\left(\sum_{j \in \underline{k}} x_{\sigma(j)} \right),$$

$$\theta_5 = f\left(\sum_{j \in \underline{m+1}} x_j \right), \quad \text{and}$$

$$-\theta_6 - \theta_7 = (-1)^{m+1-m} \sum_{\substack{\sigma \in \underline{m+1}^{\underline{m}}_{\uparrow} \\ \sigma(m-1)\neq m}} f\left(\sum x_{\sigma(j)} \right).$$

From the definition of the θ's follows

$$\begin{aligned} F_m(f, x''') &- F_m(f, x'') - F_m(f, x') \\ &= (\theta_1 + \theta_2 + \theta_5) - (\theta_1 + \theta_3 + \theta_6) - (\theta_1 + \theta_4 + \theta_7) \\ &= -\theta_1 + \theta_2 - \theta_3 - \theta_4 + \theta_5 - \theta_6 - \theta_7. \end{aligned}$$

But this last, by the equalities of the preceding paragraph, is just $F_{m+1}(f, x)$. Q.E.D.

Fix $m \in N$. Define $\phi | l_m(A, B) \to B^{(A^{\underline{m}})}$ by

$$\phi_f(x) = (1/m!)F_m(f, x) \quad \text{for all } f \in l_m(A, B) \text{ and } x \in A^{\underline{m}}.$$

THEOREM 1. *The map ϕ is an isomorphism of $l_m(A, B)$ onto* $\mathrm{sm}_m(A, B)$. *Further, for any constant function $x \in A^{\underline{m}}$, $f(x_1) = \phi(x)$.*

PROOF. Fix $f \in l_m(A, B)$. That ϕ_f is symmetric follows from the commutativity of A. To choose arbitrarily two elements of $A^{\underline{m}}$ which agree in the first $m - 1$ coordinates, we arbitrarily select an element x of $A^{\underline{m+1}}$ and consider x' and x''. The preceding lemma implies

$$\begin{aligned} \phi_f(x') + \phi_f(x'') &= (1/m!)\left[F_m(f, x') + F_m(f, x'')\right] \\ &= (1/m!)\left[F_m(f, x''') - F_{m+1}(f, x)\right] = (1/m!)F_m(f, x''') = \phi_f(x'''). \end{aligned}$$

This proves that ϕ_f is a homomorphism in the last variable. Since it is symmetric, it is a homomorphism in each variable. Thus, $\phi_f \in \mathrm{sm}_m(V)$.

The formula

$$\sum_{k \in \underline{m}} (-1)^{m-k} \binom{m}{k} k^m = m!$$

can be found, for instance, in [2]. For each constant $x \in A^{\underline{m}}$, since $\underline{m}^{\underline{k}}\uparrow$ has cardinality $\binom{m}{k}$,

$$\begin{aligned} \phi_f(x) &= \frac{1}{m!} F_m(f, x) = \frac{1}{m!} \sum_{k \in \underline{m}} (-1)^{m-k} \sum_{\sigma \in \underline{m}^{\underline{k}}\uparrow} f\left(\sum_{j \in \underline{k}} x_{\sigma(j)}\right) \\ &= \frac{1}{m!} \sum_{k \in \underline{m}} (-1)^{m-k} \binom{m}{k} f(kx_1) \\ &= f(x_1) \frac{1}{m!} \sum_{k \in \underline{m}} (-1)^{m-k} \binom{m}{k} k^m = f(x_1). \end{aligned}$$

This proves that ϕ is injective. That it is a homomorphism is clear.

We now finish the proof by showing that ϕ is surjective. Let ψ be in $\mathrm{sm}_m(A, B)$ and define $g \in B^A$ by $g(x_1) = \psi(x)$ for all constant $x \in A^{\underline{m}}$.

Claim. For all $n \in N$ and $x \in A^{\underline{n}}$,

$$F_n(g, x) = \sum_{\substack{\sigma \in \underline{n}^{\underline{m}} \\ \sigma(\underline{m}) = \underline{n}}} \psi(x_\sigma).$$

The claim is trivially true for $n = 1$. Suppose the claim is known to hold for all $k \in \underline{n}$, n an element of N. Then, for each $x \in A^{\underline{n+1}}$,

$$F_n(g, x''') = \sum_{\substack{\sigma \in \underline{n}^{\underline{m}} \\ \sigma(\underline{m}) = \underline{n}}} \psi(x_\sigma''')$$

$$= \sum_{\substack{\sigma \in \underline{n}^{\underline{m}} \\ \sigma(\underline{m}) = \underline{n}}} \psi(x_\sigma') + \sum_{\substack{\sigma \in \underline{n}^{\underline{m}} \\ \sigma(\underline{m}) = \underline{n}}} \psi(x_\sigma'') + \sum_{\substack{\sigma \in \underline{n+1}^{\underline{m}} \\ \sigma(\underline{m}) = \underline{n+1}}} \psi(x_\sigma)$$

$$= F_n(g, x') + F_n(g, x'') + \sum_{\substack{\sigma \in \underline{n+1}^{\underline{m}} \\ \sigma(\underline{m}) = \underline{n+1}}} \psi(x_\sigma).$$

It now follows from the lemma that

$$F_{n+1}(g, x) = \sum_{\substack{\sigma \in \underline{n+1}^{\underline{m}} \\ \sigma(\underline{m}) = \underline{n+1}}} \psi(x_\sigma).$$

This proves the claim. Taking n to be $m + 1$, we have

$$F_{m+1}(g, x) = \sum_{\sigma \in \varnothing} \psi(x_\sigma) = 0,$$

$\varnothing$ denoting the null set. That

$$g(kx) = \psi(kx, kx, \ldots, kx) = k^m \psi(x, x, \ldots, x) = k^m g(x)$$

for each $x \in A$ and $k \in N$ is evident. It follows that $g \in l_m(A, B)$. If we take n to be m, the symmetry of ψ yields

$$F_m(g, x) = \sum_{\substack{\sigma \in \underline{m}^{\underline{m}} \\ \sigma(\underline{m}) = \underline{m}}} \psi(x_\sigma) = m!\psi(x)$$

for each $x \in A^{\underline{m}}$. It follows that ϕ maps g onto ψ. Q.E.D.

Now let V and L be topological real linear spaces. We shall write $\mathrm{csm}_m(V, L)$ and $\mathrm{cl}_m(V, L)$ for the subsets of $\mathrm{sm}_m(V, L)$ and $\mathrm{lm}_m(V, L)$ consisting of jointly continuous and continuous mappings, respectively. We shall denote the restriction of ϕ to $\mathrm{cl}_m(V, L)$ by ϕ as well.

COROLLARY 1. *The map ϕ is an isomorphism of* $\mathrm{cl}_m(V, L)$ *onto* $\mathrm{csm}_m(V, L)$.

A standard argument using the fact that the rational numbers are dense in R shows that $\mathrm{csm}_m(V, L)$ consists of precisely the continuous symmetric m-multilinear operators from V into L.

Fix m and n in N and write E for n-dimensional real linear space with its usual topology. We shall abbreviate $\mathrm{cl}_m(E, R)$ and $\mathrm{csm}_m(E, R)$ to $l_m(E)$ and $\mathrm{sm}_m(E)$ respectively. Write Ω for the set of all $\sigma \in \{0, 1, \ldots, m\}^{\underline{n}}$ such that $\sum_{k \in \underline{n}} \sigma(k) = m$ and $\underline{n}^{\underline{m}}\downarrow$ for $\{\sigma \in \underline{n}^{\underline{m}}: \sigma \text{ is nondecreasing}\}$. For $\sigma \in \underline{n}^{\underline{m}}\downarrow$, define $\hat{\sigma}$ in Ω by $\hat{\sigma}(j) \equiv$ cardinality of $\sigma^{-1}(j)$ for all $j \in \underline{n}$. The $\hat{}$ is evidently a bijection.

Denote by $P(m; n)$ the linear space of homogeneous real polynomials of degree m, in n indeterminants. Let $t \in P(m; n)^{\underline{n}}$ be a function whose range is the set of indeterminants. Then

$$T \equiv \left\{ \prod_{j \in \underline{n}} t_j^{\sigma(j)} \colon \sigma \in \Omega \right\}$$

is a basis for $P(m; n)$. Let $b \in E^{\underline{n}}$ be a function whose range is a basis for E and write b^* for the function in $(E^*)^{\underline{n}}$ whose range is the corresponding dual basis. For each $\sigma \in \underline{n}^{\underline{m}}\downarrow$, write $b^{(\sigma)}$ for the element of $\mathrm{sm}_m(E)$ defined by

$$b^{(\sigma)}(x) \equiv \frac{1}{m!} \sum_{\substack{\tau \in m^{\underline{m}} \\ \tau(\underline{m}) = \underline{m}}} \prod_{j \in \underline{m}} b^*_{\sigma(j)}(x_{\tau(j)}) \quad \text{for all } x \in E^{\underline{m}}.$$

As is well known (see [1] for instance) $S \equiv \{b^{(\sigma)} \colon \sigma \in \underline{n}^{\underline{m}}\downarrow\}$ is a basis for $\mathrm{sm}_m(E)$. The map $\hat{}$ induces a bijection between T and S which extends uniquely to an isomorphism between $P(m; n)$ and $\mathrm{sm}_m(E)$.

The space $P(m; n)$ can be realized concretely in terms of the function b by replacing the indeterminant function with b^*. We write $P_b(m; n)$ for the linear span of the functions on E: $\prod_{j \in \underline{n}} (b_j^*)^{\sigma(j)}$, $\sigma \in \Omega$. Thus, $\hat{}$ also induces an isomorphism between $\mathrm{sm}_m(E)$ and $P_b(m; n)$. For each constant function $x \in E^{\underline{m}}$ and $\sigma \in \underline{n}^{\underline{m}}\downarrow$, we have

$$b^{(\sigma)}(x) = \frac{1}{m!} \sum_{\substack{\tau \in m^{\underline{m}} \\ \tau(\underline{m}) = \underline{m}}} \prod_{j \in \underline{m}} b^*_{\sigma(j)}(x_1) = \prod_{j \in \underline{m}} b^*_{\sigma(j)}(x_1)$$

$$= \prod_{k \in \underline{n}} \left[b_k^*(x_1) \right]^{\hat{\sigma}(k)}$$

so that, by Theorem 1, $\prod_{j \in \underline{n}} (b_j^*)^{\hat{\sigma}(j)}$ is in $l_m(E)$ and $b^{(\sigma)}$ is its image by ϕ. Hence, ϕ is just the isomorphism between $\mathrm{sm}_m(E)$ and $P_b(m; n)$ induced by b.

COROLLARY 2. *For each $b \in E^{\underline{n}}$ such that $b(\underline{n})$ is a basis, $P_b(m; n) = l_m(E)$.*

Now consider a function f defined and m times continuously differentiable on some open subset W of E. For each $p \in W$ and $x \in E$, define

$$f_p^{(m)}(x) \equiv d^m f(p + tx) / (dt)^m.$$

In coordinate-free calculus, the mth derivative $D^m f(p)$ at p is often considered to be the unique element of $\mathrm{sm}_m(E)$ such that, when $b \in E^{\underline{n}}$ is any basis function inducing on E a coordinate system,

$$\left[D^m f(p) \right](b_\sigma) = \frac{\partial^m f}{\partial b_{\sigma(1)} \partial b_{\sigma(2)} \cdots \partial b_{\sigma(m)}} (p)$$

for all $\sigma \in \underline{n}^{\underline{m}}$. In particular, for $\sigma \in \underline{1}^{\underline{m}} \subset \underline{n}^{\underline{m}}$, we have

$$\left[D^m f(p) \right](b_\sigma) = \partial^m f(p) / (\partial b_1)^m = f_p^{(m)}(b_1).$$

Since $D^m f(p)$ is basis independent and b_1 was chosen arbitrarily, this means that

$$\left[D^m f(p) \right](x) = f_p^{(m)}(x_1) \quad \text{for all constant } x \in E^{\underline{m}}.$$

From the Theorem, it now follows that $D^m f(p)$ is just the image under ϕ of $f_p^{(m)}$, the latter being perhaps more easy to compute than the former.

COROLLARY 3. *For each* $x \in E^{\underline{m}}$ *and* $p \in W$, *we have*

$$[D^m f(p)](x) = \sum_{k \in \underline{m}} (-1)^{m-k} \sum_{\sigma \in \underline{m}^k \uparrow} f_p^{(m)}\left(\sum_{j \in k} x_{\sigma(j)}\right).$$

Now suppose that f is, in addition, analytic on W. Then, for each $p \in W$, the power series expansion about p for f can be expressed

$$f(x - p) = \sum_{m=0}^{\infty} f_p^{(m)}(x - p)$$

where $f_p^{(0)}$ is defined to be the constant function with value $f(p)$. Similar formulae hold for Taylor series approximations.

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