FUNCTIONAL EQUATIONS FOR POLYNOMIALS

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Abstract. The set of all continuous symmetric multilinear forms of degree \( m \) on a real topological vector space \( V \) are shown to be in one-to-one correspondence with the family of continuous scalar-valued functions on \( V \) satisfying a certain functional equation. If \( V \) is \( n \)-dimensional, these functions are precisely those which can be represented by \( m \)-homogeneous polynomials of degree \( n \) (with respect to some basis of \( V \)).

The connection between this family of generalized polynomials and the \( m \)th derivatives of a scalar-valued function is discussed.

Denote by \( N \) the set of natural numbers and, for each \( n \in N \), by \( n \) the set \( \{ m \in N : m < n \} \). For all \( m, n \in N \), write \( n^m \uparrow \) for the family of increasing functions in \( n^m \).

Let \( A \) be an Abelian group and \( B \) a linear space over a field of characteristic 0. The binary operations in \( A \) and \( B \) will both be denoted + and, for each \( n \in N \) and \( x \in A \cup B \), \( nx \) will signify \( x + x + \cdots + x \) \( (n \text{ times}) \). For each \( m \in N \), \( sm_m(A, B) \) will be the set of all symmetric functions \( \phi : A^m \to B \) such that, whenever \( k \in m \) and \( x, y, z \in A^m \) satisfy \( x_k + y_k = z_k \) and \( x_j = y_j = z_j \) for all \( j \neq k \), then \( \phi(x) + \phi(y) = \phi(z) \).

For each \( m \in N \), define \( F_m | B^A \times A^m \to B \) by letting

\[
F_m (f, x) = \sum_{k \in m} (-1)^{m-k} \sum_{\sigma \in n^m} f \left( \sum_{j \in k} x_{\sigma(j)} \right)
\]

for all \( f \in B^A \) and \( x \in A^m \). Define \( l_m(A, B) \) to be the set of all \( f \in B^A \) such that \( f(kx) = k^n f(x) \) and \( F_{m+1}(f, y) = 0 \) for all \( k \in n \), \( x \in A \), and \( y \in A^{m+1} \). Note that \( l_1(A, B) = \text{HOM}(A, B) = sm_1(A, B) \). We shall show that \( l_m(A, B) \) and \( sm_m(A, B) \) are isomorphic for larger \( m \) as well.

For \( m \in N \) and \( x \in A^{m+1} \), we shall write \( x', x'', \) and \( x''' \) for the elements of \( A^{m+1} \) satisfying \( x_j = x_j' = x_j'' = x_j''' \) for \( j \in m+1 \) and \( x_m = x_m', x_m'' = x_m''' = x_{m+1}' \), and \( x_m''' = x_m + x_{m+1} \).

Lemma. For all \( m \in N \), \( x \in A^{m+1} \), and \( f \in B^A \),

\[
F_{m+1} (f, x) = F_m (f, x''') - F_m (f, x'') - F_m (f, x').
\]

Proof. Define
\[ \theta_1 \equiv \sum_{k \in m-1} (-1)^{m-k} \sum_{\sigma \in m^{-1}} f\left( \sum_{j \in k} x_{\sigma(j)} \right). \]

\[ \theta_2 \equiv \sum_{k \in m-1} (-1)^{m-k} \sum_{\sigma \in m^{+}\uparrow \uparrow} f\left( \sum_{j \in k} x_{\sigma(j)} \right). \]

\[ \theta_3 \equiv \sum_{k \in m-1} (-1)^{m-k} \sum_{\sigma \in m^{+} \uparrow} f\left( \sum_{j \in k} x_{\sigma(j)} \right), \]

\[ \theta_4 \equiv \sum_{k \in m-1} (-1)^{m-k} \sum_{\sigma \in m^{-1}} f\left( \sum_{j \in k} x_{\sigma(j)} \right). \]

\[ \theta_5 \equiv f\left( \sum_{j \in m} x_j \right), \quad \theta_6 \equiv f\left( \sum_{j \in m} x_j'' \right), \quad \text{and} \quad \theta_7 \equiv f\left( \sum_{j \in m} x_j \right). \]

The following equalities are evident:

\[ -\theta_1 = \sum_{k=1}^{m-1} (-1)^{m+1-k} \sum_{\sigma \in m^{-1}} f\left( \sum_{j \in k} x_{\sigma(j)} \right), \]

\[ \theta_2 = \sum_{k=2}^{m} (-1)^{m+1-k} \sum_{\sigma \in m^{+} \uparrow \uparrow} f\left( \sum_{j \in k} x_{\sigma(j)} \right), \]

\[ \theta_3 = \sum_{k=1}^{m-1} (-1)^{m+1-k} \sum_{\sigma \in m^{+} \uparrow} f\left( \sum_{j \in k} x_{\sigma(j)} \right), \]

\[ \theta_4 = \sum_{k=1}^{m-1} (-1)^{m+1-k} \sum_{\sigma \in m^{-1}} f\left( \sum_{j \in k} x_{\sigma(j)} \right), \]

\[ \theta_5 = f\left( \sum_{j \in m+1} x_j \right), \quad \text{and} \]

\[ -\theta_6 - \theta_7 = (-1)^{m+1-m} \sum_{\sigma \in m^{+} \uparrow} f\left( \sum_{j \in m} x_{\sigma(j)} \right). \]

From the definition of the \( \theta \)'s follows
\[ F_m(f, x''') - F_m(f, x''') - F_m(f, x') = \left( \theta_1 + \theta_3 + \theta_5 \right) - \left( \theta_1 + \theta_3 + \theta_6 \right) - \left( \theta_1 + \theta_4 + \theta_7 \right) = -\theta_1 + \theta_2 - \theta_3 - \theta_4 + \theta_5 - \theta_6 - \theta_7. \]

But this last, by the equalities of the preceding paragraph, is just \( F_{m+1}(f, x) \).

Q.E.D.

Fix \( m \in \mathbb{N} \). Define \( \phi \colon l_m(A, B) \to B^{(A^m)} \) by

\[ \phi_f(x) = \left( \frac{1}{m!} \right) F_m(f, x) \quad \text{for all } f \in l_m(A, B) \text{ and } x \in A^m. \]

**Theorem 1.** The map \( \phi \) is an isomorphism of \( l_m(A, B) \) onto \( sm_m(A, B) \).

**Further,** for any constant function \( x \in A^m \), \( f(x_i) = \phi(x) \).

**Proof.** Fix \( f \in l_m(A, B) \). That \( \phi_f \) is symmetric follows from the commutativity of \( A \). To choose arbitrarily two elements of \( A^m \), which agree in the first \( m - 1 \) coordinates, we arbitrarily select an element \( x \) of \( A^{m+1} \) and consider \( x' \) and \( x'' \). The preceding lemma implies

\[ \phi_f(x') + \phi_f(x'') = \left( \frac{1}{m!} \right) \left[ F_m(f, x') + F_m(f, x'') \right] = \left( \frac{1}{m!} \right) \left[ F_m(f, x''') - F_{m+1}(f, x) \right] = \left( \frac{1}{m!} \right) F_m(f, x''') = \phi_f(x''). \]

This proves that \( \phi_f \) is a homomorphism in the last variable. Since it is symmetric, it is a homomorphism in each variable. Thus, \( \phi_f \in sm_m(V) \).

The formula

\[ \sum_{k \in m} (-1)^{m-k} \binom{m}{k} k^m = m! \]

can be found, for instance, in [2]. For each constant \( x \in A^m \), since \( m^k \) has cardinality \( \binom{m}{k} \),

\[ \phi_f(x) = \frac{1}{m!} F_m(f, x) = \frac{1}{m!} \sum_{k \in m} (-1)^{m-k} \sum_{\sigma \in m^k} f \left( \sum_{j \in k} x_{\sigma(j)} \right) \]

\[ = \frac{1}{m!} \sum_{k \in m} (-1)^{m-k} \binom{m}{k} f(kx_1) \]

\[ = f(x_1) \frac{1}{m!} \sum_{k \in m} (-1)^{m-k} \binom{m}{k} k^m = f(x_1). \]

This proves that \( \phi \) is injective. That it is a homomorphism is clear.

We now finish the proof by showing that \( \phi \) is surjective. Let \( \psi \) be in \( sm_m(A, B) \) and define \( g \in B^A \) by \( g(x_1) = \psi(x) \) for all constant \( x \in A^m \).

**Claim.** For all \( n \in \mathbb{N} \) and \( x \in A^n \),

\[ F_n(g, x) = \sum_{\sigma \in n^m} \psi(x) \]

The claim is trivially true for \( n = 1 \). Suppose the claim is known to hold for all \( k \in n \), \( n \) an element of \( N \). Then, for each \( x \in A^{n+1} \),
\[ F_n(g, x'') = \sum_{\sigma \in \Pi_m} \psi(x_{\sigma''}) \]

\[ = \sum_{\sigma \in \Pi_m} \psi(x_\sigma') + \sum_{\sigma \in \Pi_m} \psi(x_\sigma'') + \sum_{\sigma \in \Pi_{m+1}^n} \psi(x_\sigma) \]

\[ = F_n(g, x') + F_n(g, x'') + \sum_{\sigma \in \Pi_{m+1}^n} \psi(x_\sigma). \]

It now follows from the lemma that

\[ F_{n+1}(g, x) = \sum_{\sigma \in \Pi_{n+1}^m} \psi(x_\sigma). \]

This proves the claim. Taking \( n \) to be \( m + 1 \), we have

\[ F_{m+1}(g, x) = \sum_{\sigma \in \emptyset} \psi(x_\sigma) = 0, \]

\( \emptyset \) denoting the null set. That

\[ g(kx) = \psi(kx, kx, \ldots, kx) = k^m \psi(x, x, \ldots, x) = k^m g(x) \]

for each \( x \in A \) and \( k \in \mathbb{N} \) is evident. It follows that \( g \in l_m(A, B) \). If we take \( n \) to be \( m \), the symmetry of \( \psi \) yields

\[ F_m(g, x) = \sum_{\sigma \in \Pi_m^m} \psi(x_\sigma) = m! \psi(x) \]

for each \( x \in A^m \). It follows that \( \phi \) maps \( g \) onto \( \psi \). Q.E.D.

Now let \( V \) and \( L \) be topological real linear spaces. We shall write \( \text{csm}_m(V, L) \) and \( \text{cl}_m(V, L) \) for the subsets of \( \text{sm}_m(V, L) \) and \( \text{lm}_m(V, L) \) consisting of jointly continuous and continuous mappings, respectively. We shall denote the restriction of \( \phi \) to \( \text{cl}_m(V, L) \) by \( \phi \) as well.

**Corollary 1.** The map \( \phi \) is an isomorphism of \( \text{cl}_m(V, L) \) onto \( \text{csm}_m(V, L) \).

A standard argument using the fact that the rational numbers are dense in \( \mathbb{R} \) shows that \( \text{csm}_m(V, L) \) consists of precisely the continuous symmetric \( m \)-multilinear operators from \( V \) into \( L \).

Fix \( m \) and \( n \) in \( \mathbb{N} \) and write \( E \) for \( n \)-dimensional real linear space with its usual topology. We shall abbreviate \( \text{cl}_m(E, R) \) and \( \text{csm}_m(E, R) \) to \( l_m(E) \) and \( \text{sm}_m(E) \) respectively. Write \( \Omega \) for the set of all \( \sigma \in \{0, 1, \ldots, m\}^n \) such that \( \sum_{k \in \sigma} \sigma(k) = m \) and \( m^n \) for \( \{ \sigma \in n^n : \sigma \text{ is nondecreasing} \} \). For \( \sigma \in n^n \), define \( \delta \in \Omega \) by \( \delta(j) \equiv \text{cardinality of } \sigma^{-1}(j) \) for all \( j \in \mathbb{N} \). The \( \hat{\cdot} \) evidently a bijection.

Denote by \( P(m; n) \) the linear space of homogeneous real polynomials of degree \( m \), in \( n \) indeterminants. Let \( t \in P(m; n)^2 \) be a function whose range is the set of indeterminants. Then
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\[ T \equiv \left\{ \prod_{\sigma \in \Omega} f^{(j)}(\sigma) : \sigma \in \Omega \right\} \]

is a basis for \( P(m; n) \). Let \( b \in E^n \) be a function whose range is a basis for \( E \) and write \( b^* \) for the function in \((E^*)^2\) whose range is the corresponding dual basis. For each \( \sigma \in n^m \), write \( b^{(\sigma)} \) for the element of \( \text{smm}(E) \) defined by

\[ b^{(\sigma)}(x) = \frac{1}{m!} \sum_{\tau \in m^m} \prod_{j \in m} b^{*}_{\sigma(j)}(x_{\tau(j)}) \quad \text{for all } x \in E^m. \]

As is well known (see [1] for instance) \( S = \{ b^{(\sigma)} : \sigma \in n^m \} \) is a basis for \( \text{smm}(E) \). The map \( \cdot \) induces a bijection between \( T \) and \( S \) which extends uniquely to an isomorphism between \( P(m; n) \) and \( \text{smm}(E) \).

The space \( P(m; n) \) can be realized concretely in terms of the function \( b \) by replacing the indeterminant function with \( b^* \). We write \( P_b(m; n) \) for the linear span of the functions on \( E \) by \( \prod_{\sigma \in \Omega} (b^{*})^{(j)}(\sigma) \), \( \sigma \in \Omega \). Thus, \( \cdot \) also induces an isomorphism between \( \text{smm}(E) \) and \( P_b(m; n) \). For each constant function \( x \in E^m \) and \( \sigma \in n^m \), we have

\[ b^{(\sigma)}(x) = \frac{1}{m!} \sum_{\tau \in m^m} \prod_{j \in m} b^{*}_{\sigma(j)}(x_{\tau(j)}) = \prod_{k \in n} \left[ (b^{*})(x_k) \right]^{(k)} \]

so that, by Theorem 1, \( \prod_{\sigma \in \Omega} (b^{*})^{(j)}(\sigma) \) is in \( l_m(E) \) and \( b^{(\sigma)} \) is its image by \( \phi \). Hence, \( \phi \) is just the isomorphism between \( \text{smm}(E) \) and \( P_b(m; n) \) induced by \( b \).

**Corollary 2.** For each \( b \in E^n \) such that \( b(n) \) is a basis, \( P_b(m; n) = l_m(E) \).

Now consider a function \( f \) defined and \( m \) times continuously differentiable on some open subset \( W \) of \( E \). For each \( p \in W \) and \( x \in E \), define

\[ f^{(m)}(x) = \left[ \frac{d^mf(p + tx)/(dt)^m}{m!} \right]. \]

In coordinate-free calculus, the \( m \)th derivative \( D^{mf}(p) \) at \( p \) is often considered to be the unique element of \( \text{smm}(E) \) such that, when \( b \in E^n \) is any basis function inducing on \( E \) a coordinate system,

\[ \left[ D^{mf}(p) \right](b) = \frac{\partial^{mf}(\sigma)}{\partial b_{(1)} \partial b_{(2)} \cdots \partial b_{(m)}}(p) \]

for all \( \sigma \in n^m \). In particular, for \( \sigma \in 1^m \subset n^m \), we have

\[ \left[ D^{mf}(p) \right](b_1) = \partial^{mf}(p)/(b_1)^m = f^{(m)}(b_1). \]

Since \( D^{mf}(p) \) is basis independent and \( b_1 \) was chosen arbitrarily, this means that

\[ \left[ D^{mf}(p) \right](x) = f^{(m)}(x_1) \quad \text{for all constant } x \in E^m. \]

From the Theorem, it now follows that \( D^{mf}(p) \) is just the image under \( \phi \) of \( f^{(m)} \), the latter being perhaps more easy to compute than the former.
Corollary 3. For each \( x \in E^m \) and \( p \in W \), we have
\[
\left[ D^m f(p) \right](x) = \sum_{k \in m} (-1)^{m-k} \sum_{\sigma \in m^k} f^{(m)}_p \left( \sum_{j \in k} x_{\sigma(j)} \right).
\]

Now suppose that \( f \) is, in addition, analytic on \( W \). Then, for each \( p \in W \), the power series expansion about \( p \) for \( f \) can be expressed
\[
f(x - p) = \sum_{m=0}^{\infty} f^{(m)}_p (x - p)
\]
where \( f^{(0)}_p \) is defined to be the constant function with value \( f(p) \). Similar formulae hold for Taylor series approximations.

References

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