SHORTER NOTES

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A SIMPLE PROOF OF RAMANUJAN'S $\psi_1$ SUM

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Abstract. We give a simple proof of the $\psi_1$ sum using basic hypergeometric functions.

The purpose of this note is to show that Ramanujan’s summation

$$i\psi_1\left(\frac{a}{b}, x\right) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(b; q)_n} x^n$$

follows immediately from the $q$-binomial theorem:

$$\sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} x^n = \frac{(ax; q)_\infty}{(x; q)_\infty} \quad [7, \text{p. 92}].$$

First, since

$$i\psi_1\left(\frac{a}{b}, x\right) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(b; q)_n} x^n + \sum_{n=1}^{\infty} \frac{(b^{-1}; q)_n}{(qa^{-1}; q)_n} \left(\frac{b}{ax}\right)^n,$$

we see that the $i\psi_1$ is an analytic function of $b$ provided $|q| < 1$, $|x| < 1$ and $|b| < |ax|$. To conclude we observe that (1) reduces to (2) whenever $b = q^m$, $m$ a positive integer.
\[ \psi_1 \left( \frac{a; q, x}{q^m} \right) = \sum_{k=1}^{\infty} \frac{(a; q)_k}{(q^m; q)_k} x^k = x^{1-m} \sum_{k=0}^{\infty} \frac{(aq^{1-m}; q)_k}{(q; q)_k} x^k \]

Hence (1) is valid in general since it holds on a convergent sequence within the domain of analyticity.

The known proofs of (1) use either tricky rearrangements of series or functional equations, see Andrews [1], [2], Andrews and Askey [4], Hahn [5] and M. Jackson [6].

Finally we note that the \( \psi_1 \) sum includes the Jacobi triple product identity, see Andrews [3, pp. 169 – 172], as a limiting case, because

\[ \sum_{n=-\infty}^{\infty} q^n z^n = \lim_{c \to 0} \psi_1 \left( \frac{-1/c; q^2, qx}{0} \right) = \left( q^2; q^2 \right)_\infty \left( -q^{-1}; q^2 \right)_\infty \left( -qx; q^2 \right)_\infty. \]

REFERENCES


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