THE UNION OF TWO HILBERT CUBES MEETING IN A HILBERT CUBE NEED NOT BE A HILBERT CUBE

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Abstract. An example is given to verify the assertion of the title.

For some time there seems to have been a great deal of interest in the following question, which we shall herein answer in the negative: If \( X = Q_1 \cup Q_2 \), where \( Q_1 \approx Q_2 \approx Q_1 \cap Q_2 \approx Q \), then is \( X \approx Q \)? Here \( Q \) denotes the Hilbert cube. The question appears to have first been raised in [1], and has appeared several times since, notably in [3] and [5]. Anderson [1] showed that the answer is yes if \( Q_1 \cap Q_2 \) is a Z-set [2] in each of \( Q_1 \) and \( Q_2 \), and, more recently, Handel [10] has shown that the answer is yes if \( Q_1 \cap Q_2 \) is a Z-set in either \( Q_1 \) or \( Q_2 \).

It should be remarked that our example involves nothing more than an observation regarding already known facts. The first part of the observation is that the construction of Eaton's "dogbone" decomposition [9] of \( E^n \) can be carried out in \( Q \), and that the resulting decomposition space \( X \) is not homeomorphic to \( Q \). (This has evidently been noted by many others; see, e.g., the remark on p. 153 of [5].) The second part of the observation is that \( X \) can be decomposed as \( Q_1 \cup Q_2 \) where \( Q_1 \approx Q_2 \approx Q_1 \cap Q_2 \approx Q \).

The fact cited as the second part of our observation is a consequence of the following technical lemma.

**Lemma.** Suppose \( M \) is a Q-manifold, \( C \) a 0-dimensional compactum, and \( F: C \times [0, 1] \to M \) an embedding such that if \( 0 < \delta < 1 \), \( F(C \times [\delta, 1]) \) is a Z-set in \( M \). Let \( G \) denote the upper semicontinuous decomposition of \( M \) whose nondegenerate elements are the members of the set \( \{ F(\{x\} \times [0, 1]) | x \in C \} \). Then \( M/G \cong M \).

**Proof.** It suffices to show that \( G \) satisfies the following version of the Bing Shrinking Criterion (cf. [6, p. 359, III]): if \( U \) is an open set containing the union of the nondegenerate elements of \( G \) and \( \varepsilon > 0 \), then there exists a homeomorphism \( h \) of \( M \) onto itself such that \( h(p) = p \) for all \( p \in M - U \) and \( \text{diam} \ h(g) < \varepsilon \) for all \( g \in G \). To verify this, suppose such a \( U \) and \( \varepsilon \) are

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Presented to the Society, March 5, 1976 under the title Some bad embeddings of Q in Q; received by the editors November 17, 1975.


Key words and phrases. Hilbert cube, dogbone decomposition, Z-set.

This research was supported by a grant from the UNC-G Research Council.

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given. Choose $\delta$ such that $0 < \delta < 1$ and $\text{diam } F([x] \times [0, 2\delta]) < \epsilon$ for all $x \in C$. Let $\mathcal{V}$ be an open cover of $U' = U - F(C \times \{0\})$ such that (1) if $x \in C$, then there exists $V \in \mathcal{V}$ such that $F([x] \times [\delta, 1]) \subset V$, and (2) if $h'$ is a homeomorphism of $U'$ onto itself which is $\text{St}^{t}(\mathcal{V})$-close to the identity, then $h'$ extends, via the identity on $M - U'$, to a homeomorphism of $M$ onto itself. Let $A = C \times (0, 1]$. Define $f': A \to U'$ by $f'(x, t) = F(x, t)$ for all $(x, t) \in A$. Define $g': A \to U'$ by

$$g'(x, t) = \begin{cases} F(x, t) & \text{if } x \in C \text{ and } 0 < t \leq \delta, \\ F\left(x, \frac{\delta t + \delta - 2\delta^2}{1 - \delta}\right) & \text{if } x \in C \text{ and } \delta < t \leq 1. \end{cases}$$

Then $f'(A)$ and $g'(A)$ are Z-sets in $U'$ and $f' \approx_{p} g'$ via a proper homotopy limited by $\mathcal{V}$. It follows from Theorem 6.1 of [4] that there exists an invertible ambient isotopy $H: U' \times [0, 1] \to U'$ such that $H_0 = \text{id}$, $H_1 f' = g'$, and $H$ is limited by $\text{St}^{t}(\mathcal{V})$. Extending $H_1$ to $M$ via the identity on $M - U'$, we obtain a homeomorphism $h$ of $M$ onto itself satisfying the requirements of the shrinking criterion.

**The example.** In [9], Eaton constructs a “dogbone” decomposition of $E^n$, $n > 3$, using a “ramified” version of Blankinship’s construction of a wild Cantor set in $E^n$ [7]. Wong [11] modified Blankinship’s construction to obtain a Cantor set $K \subset Q$ such that $\pi_1(Q - K)$ is nontrivial. Using Wong’s description, it is easy to see how to ramify the construction of $K$ as in [9] to obtain a Cantor set $L$ in $Q$ having the necessary complications for the construction of [9] to be carried out. (The formalities of the ramification process are detailed in [8, §4].) In essence this construction can be thought of as identifying a Cantor set of Cantor sets, each embedded in $Q$ as $K$ is embedded in $Q$.

Now, $Q - L$ contains a Z-set homeomorphic to $Q$ and, as previously mentioned, gluing two copies of $Q$ along such a set yields a copy of $Q$. Identifying the resulting space with $Q$, we see that we may write $Q = Q' \cup Q''$, where $Q''' = Q' \cap Q'' \equiv Q$ is a Z-set in each of $Q'$ and $Q''$, and where there exist homeomorphisms $h': Q \to Q'$ and $h'': Q \to Q''$ such that $h'(L) \cap Q''' = \emptyset = h''(L) \cap Q'''$. We may further assume that the manifolds of the “special defining sequences” [8] used in defining $h'(L)$ and $h''(L)$ fail to intersect $Q'''$.

The reader who is familiar with [9] will now have no difficulty in seeing how to obtain an embedding $H: C \times [0, 1] \to Q$, where $C$ is a Cantor set, such that

1. for each $x \in C$, $H([x] \times [0, 1])$ is a Z-set in $Q$,
2. $H(C \times \{0\}) = h'(L)$ and $H(C \times \{1\}) = h''(L)$,
3. $Q''' \cap H(C \times [0, 1]) = H(C \times \{\frac{1}{2}\})$,
4. if $0 < \delta < \frac{1}{2}$, then $H(C \times [\delta, \frac{1}{2}])$ is a Z-set in $Q'$ and $H\left(C \times \left[\frac{1}{2}, 1 - \delta\right]\right)$
is a $Z$-set in $Q^\prime$, and

(5) if $G$ is the upper semicontinuous decomposition of $Q$ whose nondegenerate elements are the members of the set $\{H(\{x\} \times [0, 1]) | x \in C\}$, then $Q/G \cong Q$.

Let $X = Q/G$ and let $\Pi: Q \to X$ denote the natural projection. Letting $Q_1 = \Pi(Q')$ and $Q_2 = \Pi(Q'')$, it follows from the Lemma and condition (4) above that $Q_1 \cong Q \cong Q_2$. But $X = Q_1 \cup Q_2$ and, by condition (3), $Q_1 \cap Q_2 = \Pi(Q''') \cong Q$, so $X$ provides the promised example.

We note that the above construction can be carried out in such a way that $H(C \times \{1\}) = C'$ is a $Z$-set in $Q'''$. Then there is a $(f-d)$ cap set in $Q'' - C'$, and it follows easily that there is a $(f-d)$ cap set in $Q_1 \cap Q_2$ which is an $\sigma$-$Z$-set in each of $Q_1$ and $Q_2$. On the other hand, we could arrange things so that $H(C \times \{1\}) = C''$ is embedded in $Q'''$ as $L$ is embedded in $Q$, with $H(C \times [0, 1])$ providing a "mismatch" between $h'(L)$ and $C''$. In this case, there will be a disk $D$ in $Q_1 \cap Q_2$ such that any disk "close" to $D$ in $Q_1 \cap Q_2$ contains a Cantor set whose complement in $Q_1$ fails to be simply connected (the argument for this is contained in [8]). It follows that in this case there does not exist a $(f-d)$ cap set in $Q_1 \cap Q_2$ which is an $\sigma$-$Z$-set in $Q_1$.

ADDED IN PROOF. J. Quinn and R. Y. T. Wong have recently shown that the union of two Keller cubes (compact convex infinite dimensional subsets of $l_2$) meeting in a Keller cube is homeomorphic to $Q$.

REFERENCES


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