THE NORMED SPACE NUMERICAL INDEX OF
C*-ALGEBRAS

TADASHI HURUYA

Abstract. Given a complex C*-algebra X, we prove that the normed space
numerical index n(X) of X is 1 or \( \frac{1}{2} \) according as X is commutative or not
commutative.

Let X be a normed linear space, X* its dual space, and B(X) the normed
algebra of all bounded linear operators on X with operator norm. Given
T ∈ B(X), the numerical range V(T), and the numerical radius v(T) of T are
defined by

\[
V(T) = \{ f(Tx): x ∈ X, f ∈ X^*, f(x) = \|x\| = \|f\| = 1 \},
\]

\[
v(T) = \sup|\lambda|: \lambda ∈ V(T)\}.
\]

The numerical index n(X) of the space X is defined by

\[
n(X) = \inf\{v(T): T ∈ B(X), \|T\| = 1\}.
\]

Standard results for n(X) are given in [1, §32].

Theorem. Let X be a complex C*-algebra. Then n(X) is 1 or \( \frac{1}{2} \) according as
X is commutative or not commutative.

Proof. If X is commutative, n(X) = 1 by [1, Theorem 32.8]. If X is not
commutative, the numerical index of X as an algebra is \( \frac{1}{2} \) by [2, Theorem 3].
Since the left regular representation of X is isometric, it follows that n(X) ≤ \( \frac{1}{2} \).
Let T ∈ B(X) with \( \|T\| = 1 \). It remains to show that v(T) ≥ \( \frac{1}{2} \). We may
assume (by embedding X in its Arens second dual) that X is unital. Let \( \epsilon > 0 \).
Choose \( y ∈ X \) with \( \|y\| = 1, \|Ty\| > 1 - \epsilon \). By [3, Theorem 1] there exist
positive real numbers \( α_1, \ldots, α_n \) with \( \sum_{j=1}^{n} α_j = 1 \) and unitary elements
\( u_1, \ldots, u_n \) of X such that \( \|y - \sum_{j=1}^{n} α_j u_j\| < \epsilon \). For some j with 1 ≤ j ≤ n
we have \( \|Tu_j\| > 1 - 2\epsilon \). Choose a state g of X such that \( |g((Tu_j)u_j^*)| > \frac{1}{2}(1 - 2\epsilon) \). Then \( (u_j^* g)(u_j) = g(u_j u_j^*) = 1 \), and so \( (u_j^* g)(Tu_j) ∈ V(T) \).
This gives v(T) ≥ \( \frac{1}{2}(1 - 2\epsilon) \). The proof is complete.

Remark. The final step in the proof is based on an idea of Crabb (see [1,
Theorem 32.9]).

Received by the editors July 30, 1976 and, in revised form, November 22, 1976.


Key words and phrases. C*-algebra, numerical index, numerical radius, numerical range.

© American Mathematical Society 1977

289
The author wishes to thank the referee for giving a simplified version of this paper.

REFERENCE


DEPARTMENT OF MATHEMATICS, FACULTY OF EDUCATION, NIIGATA UNIVERSITY, NIIGATA, JAPAN