COMPACTIFICATION BY THE TOPOLOGIST'S SINE CURVE

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Abstract. Using a compactification of the nonnegative reals whose remainder is the topologist's sine curve, results about growths of Stone-Cech compactifications are proved. For example, it is proved that if \( \beta X \) contains a nonconstant continuous image of a compact connected LOTS, then the image is contained in \( \nu X \). This extends a result of Peter Nyikos.

In this note we discuss some of the consequences of the fact that the topologist's sine curve is \( CR^+ - R^+ \) for a particular compactification \( CR^+ \) of the nonnegative real numbers. In particular, we give a new proof of the fact that if \( \beta X - X \) is path-connected, then \( X \) is pseudocompact. We also give an apparently new proof of the fact, which was communicated in a letter by Peter Nyikos, that if \( \beta X \) contains a nonconstant path then the path is contained in \( \nu X \).

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1. Preliminaries. For general definitions, see [4]. All spaces are completely regular Hausdorff. If \( X \) is any space, \( \beta X \) denotes the Stone-Cech compactification of \( X \) and \( \nu X \) denotes the Hewitt realcompactification of \( X \). A continuum is a compact connected space. \( R \) denotes the set of real numbers and \( R^+ \) denotes the set of nonnegative real numbers. The topologist's sine curve is the space \( S = S_1 \cup S_2 \) where

\[
S_1 = \{0\} \times [-1, 1] \quad \text{and} \quad S_2 = \{(x, \sin(1/x)) : 0 < x < 1\}.
\]

\( S \) is given the induced topology from \( R^2 \).

It is known that \( S \) is the remainder of some compactification of \( R^+ \) (see, for example, [4]). It is convenient for the sake of reference, however, to describe a particular compactification, which we call \( CR^+ \), such that \( CR^+ - R^+ \) is \( S \).

We observe that the minima of the function \( \sin(1/x) \) (defined on \( (0, 1] \)) are at \( x = 2/[(4n - 1)\pi] \) for \( n = 1, 2, \ldots \), and the minimum value is \(-1\). For \( n = 1, 2, \ldots \), let \( B_n = \{(x, 1/n) : 2/[(4n - 1)\pi] < x < 1\} \), so for each \( n \), \( B_n \).
is a horizontal line segment which is $1/n$ units above the x-axis. Let $C_n$ be the line segment joining the left endpoint of $B_n$ to the right endpoint of $B_{n+1}$. Formally,

$$C_n = \left\{ \left( x, \frac{\pi(1-4n)(x-1)}{n(n+1)(4n-1)\pi - 2} + \frac{1}{n+1} \right) : \frac{2}{(4n-1)\pi} \leq x \leq 1 \right\}.$$

Let $A = \cap_{n=1}^\infty (B_n \cup C_n)$. To obtain $CR^+$ we "bend" $A$ in such a way that it conforms to $S$. Let $L = \{ (x, \sin(1/x)) + (x, y) : (x, y) \in A \}$. $L$ is clearly homeomorphic to $R^+$. Let $CR^+ = S \cup L$. Then $CR^+$ is clearly a compactification of $R^+$ (where we view $R^+$ as $L$) and $CR^+ - L = S$. For $k = 1, 2, \ldots$ let $p_k = (2/[(4k-1)\pi], (1/k) - 1)$; then the $p_k$'s are the "sharp" points of $L$ (that is, the nondifferentiable points of $L$) whose $x$-coordinates are not 1.

2. Path-connectedness properties of $\beta R^+$. For the remainder of this paper let $T = [a, b]$ be any compact connected linearly ordered topological space with $a < b$.

**Lemma 2.1.** If $K$ is a locally connected continuum and $f : K \to S$ is continuous, then either $f(K) \subseteq S_1$ or $f(K) \subseteq S_2$.

**Proof.** Any locally connected continuum contained in $S$ is clearly contained in either $S_1$ or $S_2$, and local connectedness is preserved under closed continuous maps (see [2] for example).

**Theorem 2.2.** If $K$ is a nontrivial continuum contained in $\beta R^+ - R^+$, then there is a continuous function $G : K \to S$ whose range meets both $S_1$ and $S_2$.

**Proof.** Suppose $p, q \in K, p \neq q$. Let $\tilde{U}$ and $\tilde{V}$ be disjoint closed $\beta R^+$-neighborhoods of $p$ and $q$ and let $U = \tilde{U} \cap R^+, V = \tilde{V} \cap R^+$. Choose an increasing cofinal sequence $a_1 < a_2 < \ldots$ in $R^+$ such that

$$a_i \in R^+ \setminus (U \cup V) \text{ for each } i.$$

Define $g : U \cup V \to L$ by $g|[a_n, a_{n+1}] \cap U = (1, (\sin 1) + 1), g|[a_n, a_{n+1}] \cap V = p_n$. Since the sets $[a_n, a_{n+1}] \cap U$ and $[a_n, a_{n+1}] \cap V$ are disjoint and open in $U \cup V$, $g$ is well defined and continuous. $g$ extends continuously to $\tilde{g} : R^+ \to L$. $\tilde{g}$ extends continuously to $\tilde{g}^* : \beta R^+ \to CR^+$. Then $G = \tilde{g}^*|K$ is the required function.

**Definition.** If $X$ is a space and $f : T \to X$ is continuous, $f$ is called a $T$-path joining $f(a)$ to $f(b)$; its image, which is also called a $T$-path, is denoted $\tilde{T}$.

Combining Lemma 2.1 and Theorem 2.2 for the special case when $K$ is a $T$-path gives the following:

**Corollary 2.3.** (See also[1].) $\beta R^+ - R^+$ contains no nonconstant $T$-path.

**Proposition 2.4.** If $\beta R^+$ contains a nontrivial locally connected continuum $K$, $K \subseteq R^+$. 

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Proof. It suffices to show that there can be no pair \( a, b \in K \) with \( a \in R^+, b \in \beta R^+ - R^+ \). Suppose there were such a pair; let \( g \) be a continuous map from \( \beta R^+ \) to \( S \) such that \( g|R^+ \) is a homeomorphism onto \( S_2 \). Then \( g(b) \in S_1, g(a) \in S_2 \), contradicting 2.1.

3. **T-paths in** \( \beta X \). A special case of the following theorem was given in a letter by Peter Nyikos, but his proof was quite different.

**Theorem 3.1.** If \( X \) is realcompact and \( f: T \to \beta X \) is a nonconstant T-path, then \( \bar{T} \subseteq X \).

**Proof.** Suppose \( p \neq q, p, q \in \tilde{T}, p \in \beta X - X \). There is a \( g \in C(\beta X) \) such that \( 0 < g < 1, g(q) = 0, g(p) = 1 \). Since \( X \) is realcompact, there is an \( h \in C(X), h > 0 \), such that \( h \) does not extend to \( p \). Let \( F = (g|X)(h|X) \). Then \( F \) extends to \( q \) (or is already defined at \( q \)) but \( F \) does not extend to \( p \). We now view \( F \) as a function from \( X \) to \( \beta R^+ \) and let \( F^*: \beta X \to \beta R^+ \) be its Stone extension. Then \( F^*(q) = 0, F^*(p) \subseteq R^+ \) so \( F^* \) is a nonconstant T-path containing points of \( \beta R^+ - R^+ \), contradicting Corollary 2.3.

**Corollary 3.2.** If \( \tilde{T} \) is a nonconstant T-path in \( \beta X \), then \( \bar{T} \subseteq vX \).

**Proof.** \( \beta(vX) = \beta X \). Apply Theorem 3.1 to \( vX \).

**Remark.** We note that Theorem 3.1 and Corollary 3.2 hold with “nonconstant T-path” replaced with “nontrivial locally connected continuum.”

**Corollary 3.3.** If \( \beta X - X \) is path connected, or if \( \beta X \) is path connected, or if every point of \( \beta X - X \) is an element of some nonconstant T-path of \( \beta X \) for some \( T \), then \( X \) is pseudocompact.

**Proof.** By Corollary 3.2, if every point of \( \beta X - X \) is an element of a nonconstant T-path, then every point of \( \beta X - X \) is in \( vX \), that is \( \beta X = vX \).

**References**


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