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A GEOMETRICAL PROCEDURE FOR KILLING THE MIDDLE DIMENSIONAL HOMOLOGY GROUPS OF ALGEBRAIC HYPERSURFACES

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ABSTRACT. Explicit construction for decomposition of algebraic hypersurfaces into a connected sum of handles and a homological projective space is discussed. Also a connection is provided between Levine's results about the Arf invariant in the theory of knots and the computation of Arf invariant of hypersurfaces by S. Morita and J. Wood.

Morita [1] and Wood [2] have recently proved the following theorem:

THEOREM 1. *Let V_n^d be a nonsingular algebraic hypersurface of an odd dimension n and of degree d in an $(n + 1)$-dimensional complex projective space. Then there are two cases*:

(i) *If $d \not\equiv \pm 3$ (8) or $n = 1, 3, 7$, then V_n^d is decomposable into a connected sum of copies of $S^n \times S^n$, and a differentiable manifold M_n^d with middle Betti number zero.*

(ii) *If $d \equiv \pm 3$ (8) and $n \neq 1, 3, 7$, then V_n^d is decomposable into a connected sum of copies of $S^n \times S^n$ and a differentiable manifold M_n^d with middle Betti number* 2 *and Kervaire invariant* 1.

In this paper we describe an explicit geometrical construction for getting such a decomposition, from which Theorem 1 will follow. Our theorem however will not include the case $n = 4k + 1$, $d \equiv 4, 6 \pmod 8$.

In subsequent work we shall study in more detail the topological structure of M_n^d starting with the description of it obtained below, and also use the degeneration discussed here to get some information about the topology of even dimensional projective hypersurfaces.

We study a special hypersurface $V_n^d(c)$ which is a projective closure of an affine hypersurface defined by the equation

$$P_n(Z_0 \cdots Z_n) = Z_0^d + Z_1^{d-1} + Z_1 Z_2^{d-1} + \cdots + Z_{n-1} Z_n^{d-1} = C.$$

In the following theorem we discuss some properties of this hypersurface.

THEOREM 2. (i) *The hypersurface $V_n^d(c)$ is a nonsingular projective variety for $c \neq 0$, and has a single isolated singularity for $c = 0$ at the point $Z_0 = Z_1 = \cdots = Z_n = 0$.*

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(ii) *If C is sufficiently close to zero then the intersection of $V_n^d(c)$ with a disk of a small radius ε, centered at the origin, is an $(n-1)$-connected $(2n)$-manifold F_n^d and its n-dimensional Betti number b_n is given by*

$$b_n = d^{-1}\left[(d-1)^{n+2} - (d-1)\right].$$

(iii) *The characteristic polynomial $\Delta_n(t)$ of the monodromy of the isolated singularity of V_n^d can be computed by the recursive equation*

$$\Delta_{n+2}(t) = \Delta_n(t)\,\frac{t^{d(d-1)^{n+2}}-1}{t^{(d-1)^{n+2}}-1}\,\frac{t^{(d-1)^{n+1}}-1}{t^{d(d-1)^{n+1}}-1},$$

$$\Delta_1(t) = \frac{t^{d(d-1)}-1}{t^d-1}\,\frac{(t-1)}{t^{(d-1)}-1}.$$

(iv) *If n is odd then*

$$\Delta_n(1) = 1,$$

$$\Delta_n(-1) = d \quad \text{for } d \text{ odd},$$

$$\Delta_n(-1) = (d-1)^{(n+1)/2} \quad \text{for } d \text{ even}.$$

Now let T_{2n} be a $2n$-dimensional manifold which is a plumbing of two copies of a tangent bundle of the sphere S^n. Since $\Delta_n(1) = 1$, it is known that the boundary of T_{2n} is homeomorphic to S^{2n-1}. Denote by D^{2n} the disk of dimension $2n$. Then we have the following theorem.

THEOREM 3. *Define the manifold M_n^d as a gluing of $V_n^d(C) - F_n^d$ and D^{2n} when $d \not\equiv \pm 3$ (8), or $d \not\equiv 4, 6 \bmod (8)$ and $n = 4k+1$, or $n = 1, 3, 7$. Otherwise define M_n^d as a gluing of $V_n^d(C) - F_n^d$ and T_{2n}. In the first case M_n^d is a closed manifold with vanishing middle homology group. In the second case the n-dimensional Betti number of M_n^d is 2 and its Kervaire invariant is 1, except in the case when $d \equiv 4, 6 \pmod 8$ and $n = 4k+1$. In both cases the hypersurface $V_n^d(C)$ is diffeomorphic to a connected sum of M_n^d and copies of $S^n \times S^n$.*

In the case $d \not\equiv 4, 6 \pmod 8$ and $n = 4k+1$ all assertions of Theorem 1 are evidently contained in Theorem 3.

PROOF OF THEOREM 3. From the results of Levine [3] it follows that F_n^d has Kervaire invariant one if $\Delta(-1) \equiv \pm 3$ (8), and Kervaire invariant zero if $\Delta(-1) \equiv \pm 1$ (8). Hence we can derive the value of the Kervaire invariant of F_n^d by (iv) of Theorem 2. Since the boundary of F_n^d is a homotopy sphere, Wall's results [4] show the diffeomorphism type of F_n^d is determined by the Betti number b_n and the Kervaire invariant. Hence for $d \not\equiv \pm 3$ (8) or $d \not\equiv 4$, 6, and $n = 4k+1$ or $n = 1, 3, 7$, F_n^d is diffeomorphic to a connected sum of copies of $S^n \times S^n$ with removed disk, otherwise F_n^d is a connected sum of T_{2n} and copies of $S^n \times S^n$. The assertion about the Betti number of M_n^d follows from the well-known fact that the Betti number of a nonsingular hypersurface of odd dimension n and degree d is equal to $d^{-1}[(d-1)^{n+2} - (d-1)]$ (i.e., to the Betti number of F_n^d). This completes the proof of Theorem 3.

PROOF OF THEOREM 2. Assertion (i) follows by a standard computation. Next recall [5] that the polynomial $f(Z_0 \cdots Z_n)$ is weighted homogeneous of type $(W_0 \cdots W_n)$ if it can be expressed as a linear combination of monomials $Z_0^{i_0} \cdots Z_n^{i_n}$ for which

$$i_0/W_0 + i_1/W_1 + \cdots + i_n/W_n = 1. \tag{1}$$

We claim that polynomial $P_n(Z_0 \cdots Z_n)$ is weighted homogeneous of the type

$$\left[d, d-1, \ldots, \frac{d(d-1)^i}{(d-1)^i + (-1)^{i-1}}, \ldots, \frac{d(d-1)^n}{(d-1)^n + (-1)^{n-1}}\right].$$

This follows since (1) can be checked directly for every monomial of the polynomial $P(Z_0 \cdots Z_n)$.

To complete the proof of Theorem 2 we need the results of Milnor and Orlik [5] which we describe now.

To each polynomial of one variable $(t - \alpha_1), \ldots, (t - \alpha_n)$ assign the element $\langle \alpha_1 \rangle + \cdots + \langle \alpha_n \rangle$ of the group ring $Z[\mathbf{C}^*]$ of the group $\mathbf{C}^*$. We denote this element by

$$\operatorname{div}((t - \alpha_1) \cdots (t - \alpha_n)),$$

and let Λ_n denote $\operatorname{div}(t^n - 1)$. Milnor and Orlik proved

THEOREM 4 [5]. *Let $f(Z_0 \cdots Z_n)$ be a weighted homogeneous polynomial of type $(W_0 \cdots W_n)$ having an isolated singularity at the point $Z_0 = \cdots = Z_n = 0$. Then for C sufficiently close to zero the intersection of the hypersurface $f(Z_0 \cdots Z_n) = C$ with a disk of small radius ε centered at the origin, is an $(n-1)$-connected $2n$-manifold and*

(a) *the rank of the n-dimensional homology group of this intersection equals $(W_0 - 1) \cdots (W_n - 1)$;*

(b) *if $\Delta(t)$ is the characteristic polynomial of monodromy of the isolated singularity of $f(Z_0 \cdots Z_n) = 0$ then*

$$\operatorname{div} \Delta_n(t) = \left(v_0^{-1}\Lambda_{u_0} - 1\right) \cdots \left(v_n \Lambda_{u_n} - 1\right)$$

where the weights are expressed in the form $w_i = u_i/v_i$ $(i = 0, \ldots, n)$ with $u_i v_i$-mutually prime integer numbers.

We are ready to show (ii). We use the values of the weight $W_0 \cdots W_n$ which were found above, and then from (a) of Theorem 4 it follows that the expression for the rank of $H_n(F_n^d, Z)$ is

$$\begin{aligned}
\mu = (W_0 - 1) \cdots (W_n - 1) &= (d-1) \prod_{i=1}^{n} \left[\frac{d(d-1)^i}{(d-1)^i + (-1)^{i-1}} - 1\right] \\
&= (d-1) \prod_{i=1}^{n} \frac{(d-1)^{i+1} + (-1)^i}{(d-1)^i + (-1)^{i-1}} \\
&= d^{-1}\left[(d-1)^{n+2} - (d-1)\right].
\end{aligned}$$

(iii) We prove now that

$$\text{div}\,\Delta_n(t) = \sum_{i=0}^{n} (-1)^{n-i}\Lambda_{d(d-1)^i} + \sum_{i=0}^{n} (-1)^{n-i-1}\Lambda_{(d-1)^i}. \tag{2}$$

The proof of (2) is based on the following identities [5]:

$$\Lambda_{(d-1)^{n+1}}\Lambda_{d(d-1)^i} = (d-1)^i\Lambda_{d(d-1)^{n+1}},$$

$$\Lambda_{(d-1)^{n+1}}\Lambda_{(d-1)^i} = (d-1)^i\Lambda_{(d-1)^{n+1}}.$$

The inductive step follows from the next computation:

$$\begin{aligned}
\text{div}\,\Lambda_{n+1} &= \text{div}\,\Lambda_n \cdot \left(\frac{d}{(d-1)^{n+1}+(-1)^n}\Lambda_{(d-1)^{n+1}} - 1\right)\\
&= \left(\sum_{i=0}^{n}(-1)^{n-i}\Lambda_{d(d-1)^i} + \sum_{i=0}^{n}(-1)^{n-i-1}\Lambda_{(d-1)^i}\right)\\
&\quad \times \left(\frac{d}{(d-1)^{n+1}+(-1)^n}\Lambda_{(d-1)^{n+1}} - 1\right)\\
&= \sum_{i=0}^{n}(-1)^{n-i-1}\Lambda_{d(d-1)^i} + \sum_{i=0}^{n}(-1)^{n-i}\Lambda_{(d-1)^i}\\
&\quad + \left[\frac{d}{(d-1)^{n+1}+(-1)^n}\sum_{i=0}^{n}(-1)^{n-i}(d-1)^i\right]\Lambda_{d(d-1)^{n+1}}\\
&\quad + \left[\frac{d}{(d-1)^{n+1}+(-1)^n}\sum_{i=0}^{n}(-1)^{n-i}(d-1)^i\right]\Lambda_{d(d-1)^{n+1}}\\
&= \sum_{i=0}^{n}(-1)^{n-i+1}\Lambda_{d(d-1)^i}\\
&\quad + \sum_{i=0}^{n}(-1)^{n-i}\Lambda_{(d-1)^i} + (-1)^{n+1}\Lambda_{d(d-1)^{n+1}} + (-1)^i\Lambda_{(d-1)^{n+1}}\\
&= \sum_{i=0}^{n+1}(-1)^{n+1-i}\Lambda_{d(d-1)^i} + \sum_{i=0}^{n+1}(-1)^{n-i}\Lambda_{(d-1)^i}.
\end{aligned}$$

From (2) it follows:

$$\Delta_{n+1}(t) = \Delta_n^{-1}(t^{d(d-1)^{n+1}} - 1)(t^{(d-1)^{n+1}} - 1)^{-1}.$$

The recursive equation mentioned in Theorem 2 is implied by the last formula. The assertion about Δ_1 follows immediately from Theorem 4.

(iv) It can be seen from the recursive equation in (iii) that

$$\Delta_n(1) = \Delta_{n+2}(1),$$

$$\Delta_n(-1) = \Delta_{n+2}(-1) \quad \text{for } d \text{ odd, and}$$

$$(d-1)\Delta_n(-1) = \Delta_{n+2}(-1) \quad \text{for } d \text{ even.}$$

The value $\Delta_1(1)$ equals 1 and $\Delta_1(-1)$ equals d for d odd and $(d-1)$ for d even.

Thus all assertions of Theorem 2 are proved.

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