COINCIDENT PAIRS OF CONTINUOUS
SECTIONS IN PROFINITE GROUPS

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ABSTRACT. Given a profinite group $G$ and a closed subgroup $H$, we show there exist continuous sections $s: (G/H)_l \to G$ and $s': (G/H)_r \to G$ of the projections $G \to (G/H)_l$ and $G \to (G/H)_r$, onto the left and right coset spaces, respectively, such that $\text{im}(s) = \text{im}(s')$.

1. Coincident pairs of sections. Given a group $G$ and a subgroup $H$, let $(G/H)_l$ and $(G/H)_r$ denote, respectively, the left and right coset spaces of $H$ in $G$. A pair of sections

$$(G/H)_l \overset{s}{\to} G \overset{s'}{\leftarrow} (G/H)_r$$

will be called coincident if $\text{im}(s) = \text{im}(s')$. Having a coincident pair of sections is equivalent to having a subset $S$ of $G$ such that

\[ G = \bigcup_{x \in S} xH = \bigcup_{x \in S} Hx \quad \text{(disjoint union)} \]

($S = \text{im}(s) = \text{im}(s')$) or to having a 1-1 correspondence $aH \leftrightarrow Hb$ between $(G/H)_l$ and $(G/H)_r$, such that $aH \cap Hb \neq \emptyset$.

The following criterion is easy to establish (see [1]).

**Lemma 1.** There exists a coincident pair of sections for $G$ mod $H$ if and only if for every $a$ in $G$, $(H: H \cap aHa^{-1}) = (H: H \cap a^{-1}Ha)$ where, in general, $(G : H)$ denotes the cardinality of $G / H$ (left or right coset space).

Not every $G \supset H$ will admit a coincident pair of sections. We thank our colleague Peter Stebe for the following example. Let $G = \langle a, b \rangle$ be the free group on two generators $a, b$ and $H$ the subgroup generated by $a^nba^{-n}$ for all $n \geq 0$. Then $aHa^{-1} < H$ (proper subgroup), $H < a^{-1}Ha$ and $(H: H \cap a^{-1}Ha) > 1$ but $(H: H \cap aHa^{-1}) = 1$.

When $(G : H) < \infty$ there is always a coincident pair of sections. More generally, if $N(H)$ is the normalizer of $H$ and $G$ is torsion mod $N(H)$, i.e. for every $x \in G, x^n \in N(H)$ for some $n > 0$, then there exists a coincident pair of sections. This can be seen using the following criterion. Given an inner automorphism $f: G \to G$ let $H_0 = H, H_n = f^{-n}(H) \cap H_{n-1}$ for $n > 0$. If for

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every f there exists n > 0 such that \( (H_{n-1} : H_n) = (H_{n-1} : f(H_n)) \), then there exists a coincident pair of sections for \( G \) mod \( H \). This criterion follows from Lemma 1 and the observation that for any subgroups \( K, L \) of \( H \), \( (K : K \cap L) = (L : K \cap L) \) implies \( (H : K) = (H : L) \).

For finite groups \( G \), one can prove the existence of coincident pairs of sections by applying the following well-known result from combinatorics. For a proof see, e.g., [2, pp. 50–51].

**Lemma 2.** If \( A \) is a nonempty finite set and \( A = \bigsqcup_{i \in I} B_i = \bigsqcup_{i \in I} C_i \) are two partitions of \( A \) with \( |B_i| = |C_i| = r \) (\( |B| \) denotes number of elements) for all \( i \) in the finite index set \( I \), then there is a bijection \( i \mapsto j \) of \( I \) onto itself such that \( B_i \cap C_j \neq \emptyset \).

We will make use of this result later on.

2. Coincident pairs of sections in profinite groups. Let \( G \) be a profinite group and let \( U \) range over the open normal subgroups of \( G \) (for definitions and basic properties see, e.g., [3, Chapter I]). Then for any closed subgroups \( H, K \) of \( G \), \( H = \lim_{\leftarrow} HU/U \) and \( H \cap K = \lim_{\leftarrow}(HU \cap KU)/U \) (\( \lim \) denotes projective limit). Using these facts, it is not hard to prove \( (H : H \cap aHa^{-1}) = (H : H \cap a^{-1}Ha) \) for any \( a \) in \( G \). Thus, by Lemma 1, there exists a coincident pair of sections for \( G \) mod \( H \).

On the other hand, it is well known that for any closed subgroup \( H \) of \( G \) there exists a continuous section \( G/H \to G \). Our objective is to prove the following

**Theorem.** If \( G \) is a profinite group, then for any closed subgroup \( H \) of \( G \) there exists a coincident pair of continuous sections for \( G \) mod \( H \).

**Proof.** Let \( S \) be the set of all triples \( (N, s, s') \) where \( N \) is a closed normal subgroup of \( G \) and \( (G/HN)_1 \to G/N \leftarrow (G/HN)_r \) is a coincident pair of continuous sections. By taking \( N = G \) we see that \( S \neq \emptyset \).

Given \( (N, s, s') \) and \( (K, t, t') \) in \( S \), let \( (N, s, s') \leq (K, t, t') \) if \( N \supset K \) and the two squares

\[
\begin{array}{ccc}
(G/HK)_1 & \to & G/K \\
\downarrow & & \downarrow \ \ \\
(G/HN)_1 & \leftarrow & (G/HK)_r \\
\end{array}
\]

\[
\begin{array}{ccc}
(G/HK) & \leftarrow & G/K \\
\downarrow & & \downarrow \ \ \\
(G/HN) & \to & (G/HK)_r
\end{array}
\]

are commutative; the vertical arrows are projections. Clearly \( \leq \) is a partial order on \( S \). Let \( (N_i, s_i, s'_i) \) be a chain in \( S \). Put \( N = \bigcap_i N_i \). Then \( HN = \bigcap_i HN_i \).
and \( G/N = \lim_i G/N_i \). From these facts we easily get a pair \((s, s')\) such that \((N, s, s') \in \mathcal{S}\) and \((N_i, s_i, s_i') \leq (N, s, s')\) for all \(i\). We can now apply Zorn’s lemma.

Let \((N, s, s')\) be a maximal element of \(\mathcal{S}\). We want to show \(N = 1\). Suppose \(N \neq 1\). Then there exists an open normal subgroup \(V\) of \(G\) such that \(N \subset V\). Let \(N_i = N \cap V\). Then \(N_i \subset N\) and \((N : N_i) < \infty\). We will produce a pair \((t, t')\) such that \((N_i, t, t') \in \mathcal{S}\) and \((N, s, s') \leq (N_i, t, t')\). Replacing \(G/N_i\) by \(G\), we may assume \(N_i = 1\). Then \(N_i\) is finite. We have the following diagram:

\[
\begin{array}{ccc}
(G/HN)_1 & \xleftarrow{s} & (G/N)_1 \\
\downarrow & & \downarrow \\
(G/HN)_r & \xleftarrow{s} & (G/N)_r
\end{array}
\]

where \(\text{im}(s) = \text{im}(s') = S\), say, and \(p, q, q', \pi, \pi', \varphi, \varphi'\) are projections, and \(s, s'\) are continuous sections of \(\varphi, \varphi'\), respectively.

Choose an open normal subgroup \(U\) of \(G\) such that \(N \cap U = 1\) and \(HN \cap U \subset H\). With some abuse of notation we put \(aUN = p(aU), aUH = \pi(aU), HUa = \pi'(Ua), aUHN = q\pi(aU)\) and \(NHUa = q'\pi'(Ua)\) for \(a \in G\). Since \(N \cap U = 1\), \(p\) restricted to \(aU\) is a homeomorphism of \(aU\) onto \(aUN\). Also, since \(HN \cap U \subset H\), \(q\) is 1-1 on \(aUH\) and \(q'\) is 1-1 on \(HUa\). Let \(K = H \cap N\). Since \(N \) is finite, there are \(n_i \in N, i \in I\), finite, such that \(N = \bigcap_i n_i K = \bigcap_i n_i K_n\). Then for each \(A = aN \in G/N, A = \bigcap_i a_n K = \bigcap_i K_n a\). Since \(A\) is finite and \(|an_i K| = |K|\), by Lemma 2, there is a bijection \(i \mapsto j\) of \(I\) onto \(I\) such that \(an_i K \cap Kn_j a \neq \emptyset\).

Now choose \(A_1, \ldots, A_m \in S = \text{im}(s) = \text{im}(s')\) such that \(S \subset \bigcap_k A_k U\), where \(U = UN/N\). Then choose \(a_k \in G\) such that \(A_k = a_k N\) and choose a bijection \(i \mapsto j\) of \(I\) onto \(I\) depending on \(k\) such that \(a_k n_i K \cap Kn_j a_k \neq \emptyset\) and finally choose an element \(b_{kj}\) in \(a_k n_i K \cap Kn_j a_k\). Since \(s\) and \(s'\) are continuous, \(s^{-1}(A_k U)\) and \(s'^{-1}(A_k U)\) are open and clearly

\[
\begin{align*}
s^{-1}(A_k U) & \subset a_k UHN, & (G/HN)_k = \bigcap_k s^{-1}(A_k U), \\
s'^{-1}(A_k U) & \subset NHUa_k, & (G/HN)_r = \bigcap_k s'^{-1}(A_k U).
\end{align*}
\]

Now let

\[
\begin{align*}
W_{ki} &= a_k n_i UH \cap q^{-1}s^{-1}(A_k U), \\
W_{kj} &= HU n_j a_k \cap q'^{-1}s'^{-1}(A_k U).
\end{align*}
\]
Then these are open sets and
\[(G/H)_1 = \coprod_{k,i} W_{ki}, \quad (G/H)_r = \coprod_{k,j} W'_{kj}.\]

Now let \(p_{kj}^{-1} : A_k U = a_k UN \to b_{kj} U\) be the inverse of \(p\) on \(b_{kj} U\) and define functions
\[(G/H)_1 \xrightarrow{t} G \xleftarrow{t'} (G/H)_r\]
by \(t = p_{kj}^{-1}s q\) on \(W_{ki}\) and \(t' = p_{kj}^{-1}s'q'\) on \(W'_{kj}\). Clearly \(t\) and \(t'\) are continuous, \(pt = sq, pt' = s'q'\). On each \(W_{ki}\), \(q \pi t = qpp_{kj}^{-1}sq = qsq = q\). Since \(\pi(W_{ki}) \subset a_k n_i UH\) and \(q\) is 1-1 on \(a_k n_i UH\), we conclude that \(\pi t = id\), i.e. that \(t\) is a section. Similarly \(t'\) is a section.

Finally, we claim that \(im(t) = im(t')\), in fact \(t(W_{ki}) = t'(W'_{kj})\) when \(i \mapsto j\) for the given \(k\). Let \(a = t(aH), aH \in W_{ki}\). Then \(a \in bU\), where \(b = b_{kj}\), say \(a = bu, u \in U\). Write \(b = a_k n_j x = x'n_j a_k, x, x' \in K\). Since \(N \cap U = 1\), \(nu = un\) for \(n \in N\). Thus \(sq(aH) = pt(aH) = p(a) = a_k uN\) and hence \(a = t(aH) = p_{kj}^{-1}sq(aH) = p_{kj}^{-1}(a_k uN)\). But since \(a_k uN \in im(s) = im(s'), a_k uN = s'(NHa_k u) = s'q'(Ha) \in A_k U;\) since \(Ha \in H U n_j a\), this means that \(Ha \in W'_{kj}\). Therefore \(t'(Ha) = p_{kj}^{-1}s'q'(Ha) = p_{kj}^{-1}(a_k uN) = a\). This completes the proof of the theorem.

**Corollary.** There exists a homeomorphism between \((G/H)_1\) and \((G/H)_r\), such that the corresponding cosets intersect.

**References**