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COINCIDENT PAIRS OF CONTINUOUS SECTIONS IN PROFINITE GROUPS

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ABSTRACT. Given a profinite group G and a closed subgroup H, we show there exist continuous sections $s: (G/H)_l \to G$ and $s': (G/H)_r \to G$ of the projections $G \to (G/H)_l$ and $G \to (G/H)_r$ onto the left and right coset spaces, respectively, such that $\mathrm{im}(s) = \mathrm{im}(s')$.

1. **Coincident pairs of sections.** Given a group G and a subgroup H, let $(G/H)_l$ and $(G/H)_r$ denote, respectively, the left and right coset spaces of H in G. A pair of sections

$$(G/H)_l \xrightarrow{s} G \xleftarrow{s'} (G/H)_r$$

will be called *coincident* if $\mathrm{im}(s) = \mathrm{im}(s')$. Having a coincident pair of sections is equivalent to having a subset S of G such that

$$G = \coprod_{x \in S} xH = \coprod_{x \in S} Hx \quad \text{(disjoint union)}$$

($S = \mathrm{im}(s) = \mathrm{im}(s')$) or to having a 1-1 correspondence $aH \leftrightarrow Hb$ between $(G/H)_l$ and $(G/H)_r$ such that $aH \cap H_b \neq \varnothing$.

The following criterion is easy to establish (see [1]).

LEMMA 1. *There exists a coincident pair of sections for G* mod H *if and only if for every a in G, $(H: H \cap aHa^{-1}) = (H: H \cap a^{-1}Ha)$ where, in general, $(G : H)$ denotes the cardinality of G/H (left or right coset space).*

Not every $G \supset H$ will admit a coincident pair of sections. We thank our colleague Peter Stebe for the following example. Let $G = \langle a, b\rangle$ be the free group on two generators a, b and H the subgroup generated by $a^n b a^{-n}$ for all $n \geqslant 0$. Then $aHa^{-1} < H$ (proper subgroup), $H < a^{-1}Ha$ and $(H: H \cap a^{-1}Ha) > 1$ but $(H: H \cap aHa^{-1}) = 1$.

When $(G : H) < \infty$ there is always a coincident pair of sections. More generally, if $N(H)$ is the normalizer of H and G is torsion mod $N(H)$, i.e. for every $x \in G$, $x^n \in N(H)$ for some $n > 0$, then there exists a coincident pair of sections. This can be seen using the following criterion. Given an inner automorphism $f: G \to G$ let $H_0 = H$, $H_n = f^{-n}(H) \cap H_{n-1}$ for $n > 0$. If for

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every f there exists $n > 0$ such that $(H_{n-1}: H_n) = (H_{n-1}: f(H_n))$, then there exists a coincident pair of sections for $G \bmod H$. This criterion follows from Lemma 1 and the observation that for any subgroups K, L of H, $(K: K \cap L) = (L: K \cap L)$ implies $(H : K) = (H : L)$.

For finite groups G, one can prove the existence of coincident pairs of sections by applying the following well-known result from combinatorics. For a proof see, e.g., [2, pp. 50–51].

LEMMA 2. *If A is a nonempty finite set and $A = \coprod_{i \in I} B_i = \coprod_{i \in I} C_i$ are two partitions of A with $|B_i| = |C_i| = r$ ($|B|$ denotes number of elements) for all i in the finite index set I, then there is a bijection $i \mapsto j$ of I onto itself such that $B_i \cap C_j \neq \varnothing$.*

We will make use of this result later on.

2. **Coincident pairs of sections in profinite groups.** Let G be a profinite group and let U range over the open normal subgroups of G (for definitions and basic properties see, e.g., [3, Chapter I]). Then for any closed subgroups H, K of G, $H = \lim_U HU/U$ and $H \cap K = \lim_U (HU \cap KU)/U$ (lim denotes projective limit). Using these facts, it is not hard to prove $(H: H \cap aHa^{-1}) = (H: H \cap a^{-1}Ha)$ for any a in G. Thus, by Lemma 1, there exists a coincident pair of sections for $G \bmod H$.

On the other hand, it is well known that for any closed subgroup H of G there exists a continuous section $G/H \to G$. Our objective is to prove the following

THEOREM. *If G is a profinite group, then for any closed subgroup H of G there exists a coincident pair of continuous sections for $G \bmod H$.*

PROOF. Let $\mathcal{S}$ be the set of all triples (N, s, s') where N is a closed normal subgroup of G and

$$(G/HN)_l \xrightarrow{s} G/N \xleftarrow{s'} (G/HN)_r$$

is a coincident pair of continuous sections. By taking $N = G$ we see that $\mathcal{S} \neq \varnothing$.

Given (N, s, s') and (K, t, t') in $\mathcal{S}$, let $(N, s, s') \leqslant (K, t, t')$ if $N \supset K$ and the two squares

$$\begin{tikzcd}
(G/HK)_l \arrow[r, "t"] \arrow[d] & G/K \arrow[d] & (G/HK)_r \arrow[l, "t'"'] \arrow[d] \\
(G/HN)_l \arrow[r, "s"] & G/N & (G/HN)_r \arrow[l, "s'"']
\end{tikzcd}$$

are commutative; the vertical arrows are projections. Clearly $\leqslant$ is a partial order on $\mathcal{S}$. Let (N_i, s_i, s_i') be a chain in $\mathcal{S}$. Put $N = \bigcap_i N_i$. Then $HN = \bigcap_i HN_i$

and $G/N = \lim_i G/N_i$. From these facts we easily get a pair (s, s') such that $(N, s, s') \in \mathcal{S}$ and $(N_i, s_i, s_i') \leqslant (N, s, s')$ for all i. We can now apply Zorn's lemma.

Let (N, s, s') be a maximal element of $\mathcal{S}$. We want to show $N = 1$. Suppose $N \neq 1$. Then there exists an open normal subgroup V of G such that $N \not\subset V$. Let $N_1 = N \cap V$. Then $N_1 < N$ and $(N : N_1) < \infty$. We will produce a pair (t, t') such that $(N_1, t, t') \in \mathcal{S}$ and $(N, s, s') < (N_1, t, t')$. Replacing G/N_1 by G, we may assume $N_1 = 1$. Then N is finite. We have the following diagram:

$$\begin{tikzcd}
(G/H)_1 \arrow[d, "q"'] & G \arrow[l, "\pi"'] \arrow[r, "\pi'"] \arrow[d, "p"] & (G/H)_r \arrow[d, "q'"'] \\
(G/HN)_1 \arrow[d, equal] & G/N \arrow[l, "\varphi"'] \arrow[r, "\varphi'"] \arrow[d, equal] & (G/HN)_r \arrow[d, equal] \\
(G/HN)_1 \arrow[r, "s"] & G/N & (G/HN)_r \arrow[l, "s'"']
\end{tikzcd}$$

where $\mathrm{im}(s) = \mathrm{im}(s') = S$, say, and p, q, q', π, π', φ, φ' are projections, and s, s' are continuous sections of φ, φ', respectively.

Choose an open normal subgroup U of G such that $N \cap U = 1$ and $HN \cap U \subset H$. With some abuse of notation we put $aUN = p(aU)$, $aUH = \pi(aU)$, $HUa = \pi'(Ua)$, $aUHN = q\pi(aU)$ and $NHUa = q'\pi'(Ua)$ for $a \in G$. Since $N \cap U = 1$, p restricted to aU is a homeomorphism of aU onto aUN. Also, since $HN \cap U \subset H$, q is 1-1 on aUH and q' is 1-1 on HUa. Let $K = H \cap N$. Since N is finite, there are $n_i \in N$, $i \in I$ finite, such that $N = \coprod_i n_i K = \coprod_i Kn_i$. Then for each $A = aN \in G/N$, $A = \coprod_i an_i K = \coprod_i Kn_i a$. Since A is finite and $|an_i K| = |Kn_i a| = |K|$, by Lemma 2, there is a bijection $i \mapsto j$ of I onto I such that $an_i K \cap Kn_j a \neq \varnothing$.

Now choose $A_1, \ldots, A_m \in S = \mathrm{im}(s) = \mathrm{im}(s')$ such that $S \subset \coprod_k A_k \overline{U}$, where $\overline{U} = UN/N$. Then choose $a_k \in G$ such that $A_k = a_k N$ and choose a bijection $i \mapsto j$ of I onto I depending on k such that $a_k n_i K \cap Kn_j a_k \neq \varnothing$ and finally choose an element b_{kij} in $a_k n_i K \cap Kn_j a_k$. Since s and s' are continuous, $s^{-1}(A_k \overline{U})$ and $s'^{-1}(A_k \overline{U})$ are open and clearly

$$s^{-1}(A_k \overline{U}) \subset a_k UHN, \qquad (G/HN)_\Upsilon = \coprod_k s^{-1}(A_k \overline{U}),$$

$$s'^{-1}(A_k \overline{U}) \subset NHUa_k, \qquad (G/HN)_r = \coprod_k s'^{-1}(A_k \overline{U}).$$

Now let

$$W_{ki} = a_k n_i UH \cap q^{-1} s^{-1}(A_k \overline{U}),$$

$$W'_{kj} = HUn_j a_k \cap q'^{-1} s'^{-1}(A_k \overline{U}).$$

Then these are open sets and

$$(G/H)_l = \coprod_{k,i} W_{ki}, \qquad (G/H)_r = \coprod_{k,j} W'_{kj}.$$

Now let $p_{kij}^{-1}: A_k \overline{U} = a_k UN \to b_{kij} U$ be the inverse of p on $b_{kij} U$ and define functions

$$(G/H)_l \xrightarrow{t} G \xleftarrow{t'} (G/H)_r$$

by $t = p_{kij}^{-1} sq$ on W_{ki} and $t' = p_{kij}^{-1} s'q'$ on W'_{kj}. Clearly t and t' are continuous, $pt = sq$, $pt' = s'q'$. On each W_{ki}, $q\pi t = \varphi p p_{kij}^{-1} sq = \varphi sq = q$. Since $\pi t(W_{ki}) \subset a_k n_i UH$ and q is 1-1 on $a_k n_i UH$, we conclude that $\pi t = \mathrm{id}$, i.e. that t is a section. Similarly t' is a section.

Finally, we claim that $\mathrm{im}(t) = \mathrm{im}(t')$, in fact $t(W_{ki}) = t'(W'_{kj})$ when $i \mapsto j$ for the given k. Let $a = t(aH)$, $aH \in W_{ki}$. Then $a \in bU$, where $b = b_{kij}$, say $a = bu$, $u \in U$. Write $b = a_k n_i x = x' n_j a_k$, $x, x' \in K$. Since $N \cap U = 1$, $nu = un$ for $n \in N$. Thus $a = bu = a_k u n_i x = x' n_j a_k u$. Thus $sq(aH) = pt(aH) = p(a) = a_k uN$ and hence $a = t(aH) = p_{kij}^{-1} sq(aH) = p_{kij}^{-1}(a_k uN)$. But since $a_k uN \in \mathrm{im}(s) = \mathrm{im}(s')$, $a_k uN = s'(NHa_k u) = s'q'(Ha) \in A_k \overline{U}$. Since $Ha \in HUn_j a$, this means that $Ha \in W'_{kj}$. Therefore $t'(Ha) = p_{kij}^{-1} s'q'(Ha) = p_{kij}^{-1}(a_k uN) = a$. This completes the proof of the theorem.

COROLLARY. *There exists a homeomorphism between* $(G/H)_l$ *and* $(G/H)_r$ *such that the corresponding cosets intersect.*

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