A COMPLEX WHICH CANNOT BE PUSHED AROUND IN E^3

MICHAEL STARBIRD[1]

ABSTRACT. This paper contains an example of a finite complex C with triangulation T which admits two linear embeddings f and g into E^3 so that although there is an isotopy of E^3 taking the embedding f to g there is no continuous family of linear embeddings of C starting at f and ending at g. No such example can exist in E^2.

1. **Introduction.** When considering complexes PL embedded in E^n, there are at least three types of motion of the complex which one could study. The most restrictive type would be motion in which each simplex is required to remain the same size and shape at each moment. Euler's flexible surface problem [4] is an example of a question concerning this type of motion. Less restrictive is the notion of a PL isotopy in which the motion involved is a continuous family of PL embeddings. In a PL isotopy h_t ($t \in [0,1]$), at every time t there is a triangulation of the complex with respect to which h_t is a linear embedding; however, there need not be one single triangulation with respect to which each of the h_t's is linear.

Between the two types of motion mentioned above is the linear isotopy with which this paper deals. A linear isotopy of a complex is the special kind of PL isotopy in which one triangulation of the complex does serve continuously for the entire family of embeddings. (See formal definition below.) One feature of linear isotopies different from PL isotopies is that any linear isotopy of a finite complex into E^n can be extended to a linear isotopy of all of E^n [2, Theorem 3.3]. One type of question associated with linear isotopies is when the existence of an ambient isotopy of E^n which takes one linear embedding of a complex to another implies the existence of a linear isotopy of the complex between the same two embeddings.

Cairns proved [3] that if f and g are two linear embeddings of a finite complex (C,T) into E^2 so that there is an orientation-preserving homeomorphism h of E^2 with $h \circ f = g$, then there is a linear isotopy of (C,T) taking f to g. In this paper an example is presented to show that the analogous theorem for E^3 is false. In a future paper the author will prove a weakened version of Cairns' theorem in E^3, namely, where the linear isotopy in the

conclusion of Cairns' theorem is with respect to some subdivision of T. (This theorem and questions related to the example are stated at the end of this paper.)

DEFINITIONS. Let C be a complex with triangulation T. Then $h: C \to E^n$ is a *linear embedding* of C (or (C,T)) into E^n if h is an embedding which is linear on each simplex in T. A *linear isotopy* of C (or (C,T)) is a continuous family of embeddings $h_t: C \to E^n$ ($t \in [0,1]$) so that for every t, h_t is a linear embedding of C into E^n. A linear isotopy is called a *simple push* if it is fixed except on the star of one vertex. A *push* is a finite sequence of simple pushes performed one after the preceding.

2. A complex which cannot be pushed around.
In [3], Cairns proved that if f and g are two linear embeddings of a finite complex (C,T) into E^2 so that there is an orientation-preserving homeomorphism h of E^2 with $h \circ f = g$, then there is a push of (C,T) from f to g. The example in this section shows that the situation is different in E^3.

The following observations about knots in E^3 are used in establishing the properties of the example.

LEMMA 2.1. *Let K be a PL knot linearly embedded in E^3 so that for every PL homeomorphism h of E^3, $h(K)$ has at least as many edges as K has. Let v_i, v_{i+1}, and v_{i+2} be three consecutive vertices of K. Then $\text{Int}(v_i v_{i+1} v_{i+2}) \cap K \neq \varnothing$.*

PROOF. Suppose no segment of K intersects the interior of the 2-simplex $v_i v_{i+1} v_{i+2}$. If any edge $v_j v_{j+1}$ intersects the open segment $(v_i v_{i+2})$, a slight adjustment of v_i would cause $v_j v_{j+1}$ to miss the 2-simplex $v_i v_{i+1} v_{i+2}$ entirely. Therefore, we adjust K so that no edge of K intersects $\text{Int}(v_i v_{i+1} v_{i+2})$ nor intersects the open segment $(v_i v_{i+2})$. Now the broken arc $v_i v_{i+1} \cup v_{i+1} v_{i+2}$ can be pushed to the segment $v_i v_{i+2}$ along the 2-simplex $v_i v_{i+1} v_{i+2}$ keeping the other edges of the adjusted K fixed, thus embedding K with fewer edges contrary to hypothesis.

COROLLARY 2.2. *A simple closed curve J in E^3 made up of five or fewer straight edges is unknotted.*

PROOF. If J has 3 or 4 edges, it is not knotted. Let J have vertices ordered consecutively as v_1, v_2, v_3, v_4, and v_5. We assume that the vertices are in general position. Let P_{123} be the plane determined by v_1, v_2, and v_3. We assume without loss of generality that v_4 lies above P_{123}. By Lemma 2.1, if J is to be knotted, the 2-simplex $v_1 v_2 v_3$ must be pierced. Hence v_5 must lie below P_{123}. Then, however, $v_2 v_3 v_4$ is not pierced, showing that J is not knotted.

EXAMPLE 2.1. This example is a 1-complex (C,T) with two linear embeddings f and g of C into E^3 so that there is an isotopy of E^3 taking $f(C)$ to $g(C)$ and yet for which there is no push nor linear isotopy h_t ($t \in [0,1]$) of (C,T) with $h_0 = f$ and $h_1 = g$.

The example is drawn in Figure 2.1. It has twelve vertices and seventeen edges.
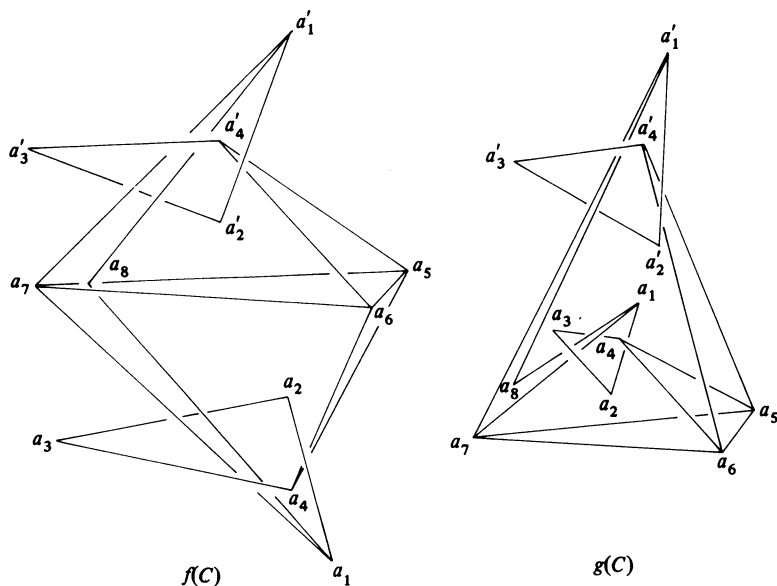

FIGURE 2.1

In order to establish the fact that this example has the properties claimed, we need to prove various facts about it.

First, we need to show that there is an isotopy of E^3 which moves $f(C)$ to $g(C)$. The isotopy is obtained by reaching down through triangle $a_5a_6a_7$ in $f(C)$ and pulling all the edges below $a_5a_6a_7$ through the triangle $a_5a_6a_7$ and arranging them as they should look in $g(C)$. Vertices $a_1', a_2', a_3', a_4', a_5, a_6$, and a_7 need not be moved in this process. Some edges will have to bend in intermediate stages of the isotopy, as we prove below. One method of demonstrating that this isotopy of E^3 exists is to build a model of $f(C)$ with string and then move it to $g(C)$ as described.

Next we show that no linear isotopy of (C, T) starts at f and ends at g.

Suppose there were a linear isotopy h_t ($t \in [0,1]$) of (C, T) so that $h_0 = f$ and $h_1 = g$. Note that under f the arc $a_1'a_8a_1$ pierces the triangle $a_5a_6a_7$ whereas under g it does not. Therefore there is an s in $[0,1]$ so that $h_s(a_1)$ or $h_s(a_1')$ lies in the 2-simplex $h_s(a_5)h_s(a_6)h_s(a_7)$. We suppose that $h_s(a_1)$ does. (The other case is equivalent because of the symmetry of C.) The strategy of the proof is to prove that h_s cannot be a linear embedding of the subcomplex of C whose vertices do not have primes.

Let P_{ijk} be the plane determined by $h_s(a_i), h_s(a_j)$, and $h_s(a_k)$.

The point $h_s(a_4)$ is either above, below, or on the plane P_{567}. It is not on P_{567} because no plane contains four vertices of a knotted simple closed curve with only six vertices such as $h_s(a_1a_2a_3a_4a_5a_7)$. Assume that $h_s(a_4)$ is below P_{567}.

By Corollary 2.2 and Lemma 2.1, we know that the 2-simplex $h_s(a_1)h_s$
$(a_5)h_s(a_7)$ must be pierced by an edge of the knot $h_s(a_1a_2a_3a_4a_5a_7)$. The only
edges available are $h_s(a_2a_3)$ and $h_s(a_3a_4)$. Assume that $h_s(a_3a_4)$ does. (See
Figure 2.2.)

Similarly the 2-simplex $h_s(a_1)h_s(a_6)h_s(a_7)$ must be pierced by $h_s(a_2a_3)$ or
$h_s(a_3a_4)$ since $h_s(a_1a_2a_3a_4a_6a_7)$ is a knot. Since it is not pierced by $h_s(a_3a_4)$, it
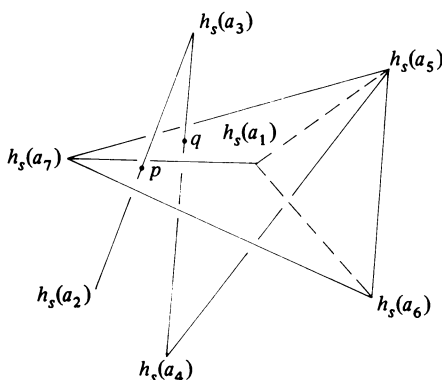must be pierced by $h_s(a_2a_3)$.

FIGURE 2.2

Let p and q be the points of intersection of the 2-simplex $h_s(a_5)h_s(a_6)h_s(a_7)$
with $h_s(a_2a_3)$ and $h_s(a_3a_4)$ respectively.

We claim that the 2-simplex $h_s(a_4)h_s(a_5)h_s(a_7)$ is pierced by neither $h_s(a_2a_3)$
nor $h_s(a_1a_2)$, thus contradicting Lemma 2.1. To see this fact note that the
segment $h_s(a_3a_4)$ lies behind the plane P_{174}. Since p is in front of P_{174}, $h_s(a_2a_3)$
must pierce P_{174} above P_{567}. Hence the segment $ph_s(a_2)$ must lie in front of
P_{174}. But the 2-simplex $h_s(a_4)h_s(a_5)h_s(a_7)$ lies behind P_{174} since it is on the
same side as q. Hence $h_s(a_2a_3)$ cannot intersect the 2-simplex $h_s(a_4)h_s$
$(a_5)h_s(a_7)$. Since $h_s(a_2)$ is in front of P_{174}, $h_s(a_1a_2)$ is also in front, hence
$h_s(a_1a_2)$ cannot intersect it either. This proves that Example 2.1 has the
properties claimed of it.

Since there is a PL homeomorphism H of E^3 with compact support taking
$f(C)$ to $g(C)$ which is linear on the simplexes in $f(C)$, Example 2.1 also
serves to prove the following theorem.

THEOREM 2.1. *There is a triangulation T of a tetrahedron σ and a linear
homeomorphism $H\colon \sigma \to \sigma$ for which there is no linear isotopy $h_t\colon \sigma \to \sigma$
($t \in [0,1]$) such that $h_0 = $ id and $h_1 = H$.*

This theorem also demonstrates a difference between the situation in E^2
and that in E^3. In fact, in E^2 the following theorem is true [2, Corollary 4.4].
If T is a triangulation of a convex PL disk P in E^2 and $H\colon P \to P$ is a linear
homeomorphism of (P,T) with $H|$Bd $P = $ id, then there is a linear isotopy $h_t\colon$

$P \to P$ ($t \in [0,1]$) so that $h_0 = \mathrm{id}$, $h_1 = H$, and for each t, $h_t |\mathrm{Bd}\ P = \mathrm{id}$.

Question 2.1. Let g_0 and g_1 be two linear embeddings of a simple closed curve J into E^3 such that there is an orientation-preserving homeomorphism h of E^3 with $h \circ g_0 = g_1$. Then can the embedding g_0 be pushed to the embedding g_1? Suppose we assume that $g_0(J)$ is unknotted?

Question 2.2. Let g be a linear embedding of a triangulated 2-sphere S. Can $g(S)$ be pushed onto the boundary of a tetrahedron?

3. **Future work.** In a future paper [6] the author will prove the following theorem.

THEOREM. *Let K be a finite complex and f and g be two PL embeddings of K into E^3 so that there is an orientation-preserving homeomorphism h of E^3 with $h \circ f = g$. Then there is a triangulation T of K and a linear isotopy (or push) h_t: $(K,T) \to E^3$ ($t \in [0,1]$) so that $h_0 = f$ and $h_1 = g$.*

REFERENCES

1. R. H. Bing, *An alternative proof that 3-manifolds can be triangulated*, Ann. of Math. (2) **69** (1959), 37–65. MR **20** #7269.

2. R. H. Bing and M. Starbird, *Linear isotopies in E^2*, Trans. Amer. Math. Soc. (to appear).

3. S. S. Cairns, *Isotopic deformations of geodesic complexes on the 2-sphere and on the plane*, Ann. of Math. (2) **45** (1944), 207–217. MR **5**, 273.

4. L. Euler, *Opera postuma*. I, Petropoli (1862), 464–496.

5. C. W. Ho, *On certain homotopy properties of some spaces of linear and piecewise linear homeomorphisms*. I,II, Trans. Amer. Math. Soc. **181** (1973), 213–243. MR **48** #1252.

6. M. Starbird, *Linear isotopies in E^3*, Proc. Amer. Math. Soc. (to appear).

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF TEXAS, AUSTIN, TEXAS 78712