DIFFEOMORPHISMS OF 3-MANIFOLDS WHICH ARE HOMOTOPY EQUIVALENT TO S¹

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Abstract. Let h be a diffeomorphism of a 3-manifold M which is homotopy equivalent to the 1-sphere. Suppose that the collection of positive iterates of h has compact closure in the space of smooth mappings of M into itself and suppose that this closed set generated by h is not a group. Necessary and sufficient conditions are given that another diffeomorphism g be topologically equivalent to h.

Let g and h be diffeomorphisms of a smooth manifold M onto itself; g and h are topologically equivalent if there exists a homeomorphism f of M onto itself such that g = f⁻¹hf. Let D(M) be the space of smooth mappings of M into itself with the fine C¹-topology [9]. Let Γ(h) be the closure of {hⁱ: i > 0} in D(M) and let Γ₀(h) = Γ(h) - {hⁱ: i > 0}. With respect to composition, Γ(h) is a topological semigroup with the topology induced from D(M). If Γ(h) is compact and Γ(h) is not contained in Γ₀(h), then Γ₀(h) is a topological group [11]. In this paper, we consider the problem of determining necessary and sufficient conditions in order that two diffeomorphisms h and g be topologically equivalent when Γ(h) and Γ(g) are compact. We shall not consider the case that Γ(h) ⊆ Γ₀(h) which reduces to the well-studied case of smooth actions of compact Lie groups on manifolds. Let πₘ be the identity element of Γ₀(h) and let I(h) = image πₘ. It has been shown that πₘ:M → I(h) is almost a “vector bundle” (cf. Proposition 2). Our main result is the following.

THEOREM 1. Let M be an open 3-dimensional manifold which is homotopy equivalent to the 1-sphere S¹. Let h and g be diffeomorphisms of M onto itself such that Γ(h) and Γ(g) are compact, Γ(h) ⊆ Γ₀(h), and Γ(g) ⊆ Γ₀(g). h and g are topologically equivalent if and only if (1) there exists a homeomorphism k of M onto itself such that k(I(h)) = I(g); (2) h|M - I(h) is topologically equivalent to k⁻¹g|M - I(g); (3) if (h|M - I(h))ₘ is the homomorphism of Hₘ(M - I(h)) induced by h, then (h|M - I(h))ₘ = (k⁻¹g|M - I(h))ₘ.

For arbitrary smooth manifolds, the problem of classifying the topological type of diffeomorphisms h when h|M is the identity was studied by the author [3]. If h is not smooth, then I(h) could be a wildly embedded 1-sphere
in $M$ [5]. Since many of the techniques used in proving Theorem 1 are in [3], we will appeal often to this paper and only provide sketches of some proofs.

Let $M$ and $h$ be as in Theorem 1. From [3] and [4], we have the following propositions.

**Proposition 2.** \( \pi_h : M \to I(h) \) is a smooth map of constant rank $r$, $0 < r < 3$; $I(h)$ is a smooth submanifold of $M$ of dimension $r$ whose inclusion into $M$ is a homotopy equivalence. For each $x \in I(h)$, $\pi_h^{-1}(x)$ is homeomorphic to $\mathbb{R}^{3-r}$, $(3-r)$-dimensional Euclidean space.

**Proposition 3.** Let $\rho : M \to O(h)$ be the natural projection of $M - I(h)$ onto the orbit space of $h \mid M - I(h)$. $O(h)$ is a 3-dimensional manifold and $\rho$ is a covering map.

The author is unable to show, in general, that $\pi_h : M \to I(h)$ is a locally trivial vector bundle; however, this will turn out to be true in the particular case with which we are working.

**Proposition 4.** If $I(h)$ is compact, then $\pi_h : M \to I(h)$ is a locally trivial vector bundle.

**Proof.** Let $\alpha : N \to I(h)$ be a normal vector bundle of $I(h)$ in $M$. Note that we can choose $N$ such that $\alpha = \pi_h \mid N$ [7]. Pick a Riemannian metric for this bundle and let $\Sigma$ be the sphere bundle of radius 1. There exists $p > 0$ such that $\Sigma \cap h^{\rho}(\Sigma) = \emptyset$ for all integers $i$ (see [6] or use Proposition 3). Note that if $D$ is the disk bundle bounded by $\Sigma$, then $M = \bigcup_{i} h^i(D)$. Let $D_i$ denote the disk bundle of radius $i$. By using the uniqueness theorem of normal disk bundles, one can construct a homeomorphism $\phi : N \to M$ such that $\phi(D_i) = h^{-\rho}(D)$ for all positive integers $i$ and such that $\pi_h \phi = \alpha$. Henceforth, let $\sigma : \Sigma \to I(h)$ be the associated sphere bundle of $\pi_h : M \to I(h)$ if $I(h)$ is compact; if $I(h)$ is noncompact, choose $\Sigma$ as in the proof of Proposition 4.

If $h \mid I(h)$ is the identity, then one can define a map $\pi_h' : O(h) \to I(h)$ such that $\pi_h = \pi_h' \rho$; this construction was invaluable in [3].

Since $h \mid I(h) = h \pi_h \mid I(h)$ and since $h \pi_h \in \Gamma_0$, we can identify $\Gamma_0$ with the compact group “generated” by $h \mid I(h)$. Hence $\Gamma_0$ is a compact Lie group. From classical transformation group theory, we have the following.

**Proposition 5.** $\Gamma_0$ is periodic or is isomorphic to either $S^1$ or $S^1 \times \mathbb{Z}_2$.

**Proposition 6.** There exists an embedding $\lambda : \Sigma \to M - I(h)$ such that $\pi_h \lambda = \alpha$, $h^i \lambda(\Sigma) \cap \lambda(\Sigma) = \emptyset$, for all integers $i$, and $\lambda$ is a homotopy equivalence.

**Proof.** Case 1. $\Gamma_0$ is periodic with period $p$. Let $\pi' = \lim_{i \to +\infty} h^{pi}$; there exists an induced map $\pi'' : O(h^p) \to I(h)$ such that $\pi' = \pi'' \rho'$ where $\rho' : M - I(h) \to O(h^p)$ is the natural projection. Let $I_0(h)$ be the orbit space of
diiffeomorphisms of 3-manifolds

By using the fibration $\pi'''$ and [1], we can show the following.

**Proposition 7.** If $\Gamma_0$ is periodic, then $\pi_h: M \to I(h)$ is a locally trivial vector bundle.

**Case 2.** $\Gamma_0$ is not periodic and dimension $I(h) = 2$. Let $\rho: I(h) \to I_0(h)$ be the natural projection of $I(h)$ onto the orbit space with respect to the action of $\Gamma_0$ restricted to $I(h)$. From classical transformation group theory, $I_0(h)$ is homeomorphic to either $\mathbb{R}$ or $[0, + \infty)$. In the latter case, we have one exceptional orbit which corresponds to 0.

Note that there exists a smooth proper map $\rho_2: O(h) \to I_0(h)$ such that $\rho_2 \rho = \rho_1 \pi_h: M - I(h) \to I_0(h)$. Note that $\rho_2$ has rank 1 at each point of $O(h)$ except at the points which correspond to the possible exceptional orbit. Hence $\rho_2$ is a locally trivial fibration except over the possible exceptional orbit. It is now easy to find an embedding of $\Sigma$ into $O(h)$ whose lifting to $M - I(h)$ is the desired embedding.

**Proposition 8.** If $\Gamma_0$ is not periodic and dimension $I(h) = 2$, then $\pi_h: M \to I(h)$ is a locally trivial vector bundle.

**Case 3.** $\Gamma_0$ is not periodic and dimension $I(h) = 1$. Note that in this case $O(h)$ is compact. By [12], there exists a surface $T \subseteq O(h)$ such that $i_*(\pi_1(T)) = \pi_*(\pi(M - I(h)))$. We may assume that $T$ is smoothly embedded in $O(h)$.

Let $x \in I(h)$ and consider $\rho|_{\pi_h^{-1}(x) - \{x\}}$. Since $\Gamma_0$ is not periodic, $\rho|_{\pi_h^{-1}(x) - \{x\}}$ is a 1-1 immersion of $\pi_h^{-1}(x) - \{x\}$ into $O(h)$. By using the implicit function theorem, one can show that $O(h)$ is foliated by $\mathcal{F} = \{\mathcal{F}_x = \rho(\pi_h^{-1}(x) - \{x\}): x \in I(h)\}$. We may assume that $T$ is transverse regular with respect to this foliation. If we intersect $T$ with leaves of $\mathcal{F}$, we get on $T$ a family of curves with singular points. Since the inclusion of $T$ can be lifted to $M - I(h)$, $T \cap \mathcal{F}_z$ is compact for each $z$. We may assume that each leaf contains at most one singular point. The following procedure was motivated by S. P. Novikov’s proof of Theorem 6.1 of [10]. There are two types of singular points: “central” and “saddle”. Let $x$ be a central singular point;
close to \( x \), the simple closed curves of the intersection of \( T \) with the leaves of \( \mathcal{F} \) must be null-homotopic on their respective leaves. In fact, if \( \tau \) is a component of \( T \cap \mathcal{F} \), which is a simple closed curve bounding a disk on \( T \), then \( \tau \) must be null-homotopic on \( \mathcal{F} \) since the immersion of each leaf induces a monomorphism on the fundamental group. There exists a curve \( \xi \) in \( T \cap \mathcal{F} \) for some \( w \) such that \( \xi \) contains a simple closed curve \( \xi_0 \) which is the boundary of a disk \( D \) on \( T \) such that \( D \) contains \( x \) in its interior and \( D \) contains only one other singular point \( y \) on its boundary. \( \xi_0 \) bounds a disk \( E \) on its leaf. Note that \((\text{int } E) \cap T\) is either empty or consists of a finite number of simple closed curves \( \xi_1, \ldots, \xi_l \), \( \xi_i \) bounds a disk \( D_i \) on \( T \); suppose that \( D_i \) is contained in no \( D_j \) for \( i \neq j \). On a leaf \( \mathcal{F}_{w(0)} \) close to the leaf containing \( E \), let \( \xi'_i \) be the component of the intersection of \( T \) with \( \mathcal{F}_{w(0)} \) which is close to \( \xi_i \), \( \xi'_i \) bounds a disk \( E_i \) on \( \mathcal{F}_{w(0)} \). Replace \( T \) by \((T - D'_j) \cup E_i \) where \( D'_j \) is the cell on \( T \) bounded by \( \xi'_i \). Hence we may assume that \( T \cap (\text{int } E) = \emptyset \); note that each \( D_i \) contains at least one central singular point. Let us refer to the \( E_i \)'s as singular disks. We may assume that the \( E_i \)'s contain no singular points. Fix some \( E_i \); we can find \( \eta \) in \( T \cap \mathcal{F}_w \) such that \( \eta \) contains a simple closed curve \( \eta_0 \) which is the boundary of a disk \( G_i \) on \( T \) such that \( G_i \) contains \( E_i \) in its interior and \( G_i \) contains only one singular point on its boundary. Replace \( G_i \) by the disk \( G_i \) which is contained on the leaf which contains \( \eta_0 = \text{bdry } \mathcal{F}_i \). We may have to do some alterations to \( T \) as above to remove other intersections of \( T \) with \( G_i \). Note that the number of singular disks we obtain is never greater than the original number of central singular points. Hence, we may assume that the intersection of \( T \) with the leaves of \( \mathcal{F} \) contains no central singular points and each singular disk contains a saddle singular point in its boundary. Let \( D \) be a singular disk with saddle point \( x \); note that \( D \) has a product neighborhood \( N \times \mathbb{R} \) in \( O(h) \) such that for each \( t \in \mathbb{R} \), \( N \times \{t\} \) lies on a leaf. Hence we can tilt \( D \) slightly along this product structure keeping \( x \) fixed so that \( x \) is no longer a singular point. Hence we may assume that the intersection of \( T \) with the leaves of \( \mathcal{F} \) contains no singular disks or central singular points.

The above cutting and pasting procedures do not change the homotopy class of the inclusion of \( T \) into \( O(h) \). Hence the inclusion of \( T \) into \( O(h) \) can be lifted to an embedding \( \phi \) of \( T \) into \( M - I(h) \); note that \( \pi_h: \phi(T) \to I(h) \) is a smooth function whose critical points correspond to the image under \( \phi \) of the singular points of the intersection of \( T \) with the leaves of \( \mathcal{F} \). Let \( \tilde{\pi}_h: \phi(T) \to \mathbb{R} \) be the induced map of the covering space of \( \phi(T) \) which is the pullback of the universal covering of \( I(h) \). Note that \( \tilde{\pi}_h \) is a Morse function. If there is a singular point in the intersection of \( T \) with the leaves of \( \mathcal{F} \), then it follows from Morse theory [8] that some component of \( \phi(T) \) has infinite genus. But no covering of the torus or the Klein bottle has this property. Hence \( T \) meets the leaves of \( \mathcal{F} \) in simple closed curves. It is easily verified that if \( x \in I(h) \), then \( \phi(T) \cap \pi_h^{-1}(x) \) is connected. We leave to the reader the construction of the map \( \lambda \) from \( \Sigma \) onto the image of \( \phi \). This completes the proof of Proposition 6.
We now proceed as in [3] to finish the proof of Theorem 1. We will sketch a proof indicating some of the minor changes which are needed.

Choose a Riemannian metric for the bundle \( \pi_h \): \( M \to I(h) \) and let \( \Sigma_t \) denote the sphere bundle of those elements of \( M \) of norm \( t, t > 0 \). Let \( \beta: \Sigma \times (0, +\infty) \to M - I(h) \) be a diffeomorphism such that \( \beta(\Sigma \times (t)) = \Sigma_t \), and \( \pi_h \beta(s, t) = \sigma(s) \). By Proposition 1.3 of [3] we can find a homeomorphism \( \gamma \) of \( M \) onto itself such that \( \gamma(\lambda(\Sigma)) = \Sigma_2 \), \( \gamma(h(\lambda(\Sigma))) = \Sigma_1 \), \( \pi_h \gamma = \pi_h \), and the induced homomorphism of \( H_1(M - I(h)) \) is the identity. Define \( \delta: \Sigma \to \Sigma \) by \( \beta^{-1} \gamma \beta^{-1} \beta(x, t) = (\delta(x), 1) \). Note that \( \delta \) is a homeomorphism such that \( \sigma \delta = \sigma \). Define \( h_1 \) on \( M \) by

\[
h_1(z) = \begin{cases} 
  h(z) & \text{if } z \in I(h), \\
  \beta(\delta(x), t/2) & \text{if } z = \beta(x, t).
\end{cases}
\]

Using the same proof as the proof of Proposition 1.6 of [3], we can show the following.

**Proposition 9.** \( h \) and \( h_1 \) are topologically equivalent.

Suppose that \( k_0 \) is a homeomorphism of \( I(h) \) onto itself; one can easily show that \( k_0 \) can be extended to a homeomorphism \( k \) of \( M \) onto itself such that the induced homomorphisms on \( H_*(M - I(h)) \), \( (h|I(h))_* \) and \( (k^{-1}h_0|M - I(h))_* \), are the same. Let \( k \) be the homeomorphism given in the hypotheses of Theorem 1. Hence, we may assume that \( h|I(h) = k^{-1} \), \( |(h) \). Let \( \lambda, \gamma, \) and \( \delta \) be the analogues of \( \lambda, \gamma \), and \( \delta \), respectively, for \( k^{-1} \). Since \( \gamma: H_*(M - I(h)) \to H_*(M - I(h)) \) is the identity isomorphism, \( \delta \) and \( \delta \) induce the same isomorphism on \( H_*(\Sigma) \). If the dimension of \( I(h) \) is 2, then, since \( \sigma \delta = \sigma \), this implies that \( \delta = \delta \) and the theorem follows from Proposition 9.

Suppose that the dimension of \( I(h) \) is 1. Let us first consider the case that \( M \) is homeomorphic to \( S^1 \times \mathbb{R}^2 \). Choose simple closed curves \( s_1, s_2 \subseteq \Sigma \) such that \( s_1 \) is a fibre and \( \sigma|s_2 \) is a homeomorphism. Note that \( \delta_*(s_1) = \pm [s_1] \) and \( \delta_*(s_2) = r[s_1] + [s_2] \) where \( r \) is an integer and \( [s] \) denotes the homology class of \( s \).

Suppose that \( h \) and \( h|I(h) \) are orientation-preserving. Hence \( \delta_*(s_1) = [s_1] \) and \( \delta_*(s_2) = r[s_1] + [s_2] \); note that \( g \) and \( g|I(g) \) are orientation-preserving. Let \( \Delta \) be an orientation-preserving homeomorphism of \( \Sigma \) onto itself such that \( \sigma \Delta = h^{-1} \sigma \) and \( \delta_*(s_2) = [s_2] \); note that \( \sigma \Delta = \sigma \). Let \( \text{Aut}^+S^1 \) be the space of orientation-preserving homeomorphisms of \( S^1 \) onto itself and let \( \text{Emb}(w, S^1) \) be the space of embeddings of a point \( w \in S^1 \); give both these spaces the compact-open topology. Consider \( R: \text{Aut}^+S^1 \to \text{Emb}(w, S^1) \) which is defined by \( R(\phi) = \phi(w) \); the induced homomorphism \( \pi_1 \text{Aut}^+S^1 \to \pi_1 \text{Emb}(w, S^1) \) is an isomorphism [2]. \( \pi_1 \text{Emb}(w, S^1) \) is isomorphic to the integers; let \( R_+ \) be the composition of these isomorphisms. We may assume that \( R_+([\text{identity}]) = 1 \). Note that an element of \( \pi_1 \text{Aut}^+S^1 \) may be represented by a homeomorphism \( f \) of \( \Sigma \) onto itself such that \( \sigma f = \sigma \) and that
\( f_\ast([s_2]) = R_\ast([f])[s_1] + [s_2]. \) Since \( R_\ast[\delta \Delta] = R_\ast[\delta' \Delta], \) we can find an isotopy \( \Phi_t, t \in [0, 1], \) of \( \Sigma \) onto itself such that \( \Phi_0 = \delta \Delta, \ Phi'_1 = \delta' \Delta \) and \( \sigma \Phi'_t = \sigma \) for all \( t. \) The remainder of the argument is given in the proof of Theorem 1.7 in [3]; we give a brief sketch. Define \( \Phi: \Sigma \times [1, 2] \to \Sigma \times [1, 2] \) by \( \Phi(x, t) = (\Phi_{-1}^{-1} \Delta^{-1} \Phi^{-1}(x), t) \) and define \( F: \Sigma \times (0, +\infty) \to \Sigma \times (0, +\infty) \) by \( F(x, t) = \beta^{-1} h_1 \beta \Phi(\delta^{-q}(x), 2qt) \) where \( q \) is the unique integer such that \( 1 < 2qt < 2. \) \( F \) is a homeomorphism such that \( \alpha \Phi_1 F = \alpha \Phi_1 \) where \( \Phi_1 \) is the projection of \( \Sigma \times (0, +\infty) \) onto \( \Sigma. \) One can show that \( F^{-1} \beta^{-1} h_1 \beta F(x, t) = (\delta'(x), t/2). \) If we let \( g_1 \) be the analogue of \( h_1 \) for \( k^{-1} g_k \) and define \( F' \) on \( M \) by

\[
F'(z) = \begin{cases} 
  z, & z \in I(h), \\
  \beta F \beta^{-1}(z), & z \in I(h),
\end{cases}
\]

then \( F^{-1} h_1 F' = g_1 \) and the theorem follows from Proposition 9.

If \( h \) and \( h \mid I(h) \) are not orientation-preserving then the only changes in the above proof are that one may have to consider \( \text{Aut}^{-1} \Lambda, \) the space of orientation-reversing homeomorphisms of \( S^1 \) and/or one may need \( \Delta \) to be orientation-reversing and, thus, \( \Delta_\ast([s_2]) = -[s_2]. \)

Now suppose that \( M \) is the twisted bundle over \( S^1. \) Choose \( s_1, s_2 \subseteq \Sigma \) as above; note that \( \Delta_\ast([s_1]) = [s_1] \) and \( \Delta_\ast([s_2]) = r[s_1] \pm [s_2] \) where \( r \in Z_2. \)

Let \( \text{Aut} \Sigma \) be the set of homeomorphisms \( f \) of \( \Sigma \) onto itself such that \( af = \sigma \) and let \( \text{Sect} \Sigma \) be the set of sections of \( \Sigma; \) give both sets the compact-open topology. Fix a section \( \xi \in \text{Sect} \Sigma; \) define \( T: \text{Aut} \Sigma \to \text{Sect} \Sigma \) by \( T(f) = f \circ \xi. \) The proof of the following is straightforward.

**Proposition 10.** \( T \) induces a bijection from \( \pi_0(\text{Aut} \Sigma) \) onto \( \pi_0(\text{Sect} \Sigma). \) The natural map of \( \pi_0(\text{Sect} \Sigma) \) into the free homotopy classes of mappings of \( I(h) \) into \( \Sigma, \{I(h), \Sigma]\), is one-to-one.

But there exists a bijection from \( \{I(h), \Sigma\} \) onto \( H_1(\Sigma) \) which carries the image of \( \pi_0(\text{Sect} \Sigma) \) onto the torsion subgroup. We can now proceed as above to complete the proof of Theorem 1.

**References**


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