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mod p WU FORMULAS FOR THE STEENROD ALGEBRA AND THE DYER-LASHOF ALGEBRA[1]

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ABSTRACT. Formulas for the polynomials $\mathcal{P}^i(c_j)$ and $Q^i(a_j)$ in $H^*(BU; Z_p)$ and $H_*(BU; Z_p)$, analogous to Wu's formulas for $\mathrm{Sq}^i(w_j)$, are given.

0. **Introduction.** The actions of the Steenrod and Dyer-Lashof algebras on $H^*(BU; Z_p)$ and $H_*(BU; Z_p)$ are covered by Hopf algebra actions on $H^*(BU; Z)$ and $H_*(BU; Z)$. As a consequence, the extra information stored in the integral, rather than mod p, Newton formulas can be extracted by applying the operations and using Cartan's formulas.

As algebras,

$$H^*(BU; Z) \cong Z[c_1, c_2, \ldots, c_n, \ldots],$$
$$H_*(BU; Z) \cong Z[a_1, a_2, \ldots, a_n, \ldots]$$

if c_i is the ith Chern class and a_i is the dual of c_1^i in the basis dual to the Chern classes, if $H_*(BU; Z)$ is identified with $\mathrm{Hom}(H^*(BU; Z), Z)$. The same notation will be used for the Z_p reduction of these classes.

If $\{\mathcal{P}^i\}_{i\geq 0}$ and $\{Q^i\}_{i\geq 0}$ are the reduced Steenrod powers and the Dyer-Lashof operations, the polynomials $\mathcal{P}^r(c_s)$ and $Q^r(a_s)$ will be given in closed form.

An algorithm for the computation of $Q^i(a_j)$ is given in Kochman [1], and uses Nishida relations as we use lifting to Z actions. No closed formulas are given except for a handful of special instances at the prime 2. S. Priddy [3] and D. Moore [2] have computed $Q^i(a_j)$ in closed form for the prime 2 and odd primes respectively. It appears to be very difficult to establish compatibility with our results, at the prime 2, as the methods and resultant formulas are quite different. In special cases, e.g. coefficients of $a_r a_s$, or $a_r a_s a_t$, the reader should have no difficulty comparing with success. In most cases, Priddy's formulas appear to be somewhat easier to use than ours. We have not yet seen Moore's.

For properties of the algebras of operations, which are quite well known, see Kochman [1] and Steenrod and Epstein [6].

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In a companion note (Shay [4]) we will give analogous formulas in bases dual to the bases of monomials, and in bases appropriate to the Quillen decomposition of the Hopf algebras. The method of proof is not via change of basis formulas, which are not available. Professor Frank Peterson has (private communication) an algorithm for computing $\mathcal{P}^r(c_s)$ which involves a hybrid of the Milnor basis, $\{s_w\}_{w \in \Pi_1^* Z^+}$, and the Chern class monomials, and which may be more efficient for computing the value of monomial coefficients in some individual cases.

For notational convenience, p will be an odd prime throughout. A closing remark will indicate the adjustments to be made at the prime 2.

1. Let

$$\tilde{\mathcal{A}}^{\mathrm{ev}} \equiv Z\langle \tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \ldots, \tilde{\mathcal{P}}_i, \ldots \rangle; \qquad \tilde{\mathcal{P}}_i \to \sum_{j+k=i} \tilde{\mathcal{P}}_j \otimes \tilde{\mathcal{P}}_k$$

be a free associative Hopf algebra on the indicated divided power sequence ($\tilde{\mathcal{P}}_0 = 1$). Let

$$\tilde{R} \equiv Z\langle \tilde{Q}^1, \tilde{Q}^2, \ldots, \tilde{Q}^i, \ldots \rangle; \qquad \tilde{Q}^i \to \sum_{j+k=i} \tilde{Q}_j \otimes \tilde{Q}_k$$

be defined analogously. Clearly $\tilde{\mathcal{A}}^{\mathrm{ev}} \otimes Z_p$ mod Adem relations $\cong \mathcal{A}^{\mathrm{ev}}$, the sub-Hopf algebra of the Steenrod algebra generated by the reduced powers, and $\tilde{R} \otimes Z_p$ mod Adem relations $\cong R$, the Dyer-Lashof algebra.

THEOREM 1. (i) *There is an action of* $\tilde{\mathcal{A}}^{\mathrm{ev}}$ *on* $H^*(BU; Z)$ *covering the action of* $\mathcal{A}^{\mathrm{ev}}$ *on* $H^*(BU; Z_p)$; (ii) *there is an action of* $\tilde{R}$ *on* $H_*(BU; Z)$ *covering the action of* R *on* $H_*(BU; Z_p)$.

PROOF. Identify $H^*(BU; Z)$ with an algebra of symmetric functions by identifying c_i with $\sigma_i(x_1, x_2, \ldots, x_n, \ldots)$, the ith elementary symmetric function, in $Z[[x_1, x_2, \ldots, x_n, \ldots]]$. Define $\tilde{\mathcal{P}}^0(1) \equiv 1$, $\tilde{\mathcal{P}}^j(1) \equiv 0$ if $j > 1$, $\tilde{\mathcal{P}}^0(x_i) \equiv x_i$, $\tilde{\mathcal{P}}^1(x_i) \equiv x_i^p$, $\tilde{\mathcal{P}}^j(x_i) = 0$ if $j > 1$. Extend to an action of $\tilde{\mathcal{A}}^{\mathrm{ev}}$ on $Z[[x_1, x_2, \ldots, x_n, \ldots]]$ by Cartan's formulae and linearity. A trivial computation shows that

$$\tilde{\mathcal{P}}^j(x_1^i + x_2^i + \cdots + x_n^i + \cdots)$$
$$= \binom{i}{j}(x_1^{i+j(p-1)} + x_2^{i+j(p-1)} + \cdots + x_n^{i+j(p-1)} + \cdots).$$

Since the symmetric powers generate the symmetric functions over Q, and the symmetric functions are rationally closed in $Z[[x_1, x_2, \ldots, x_n, \ldots]]$, an action of $\tilde{\mathcal{A}}^{\mathrm{ev}}$ on $H^*(BU; Z)$ is defined. It is a mild exercise to show that multiplicative and comultiplicative Cartan formulae are satisfied. The former should merely be proven for products of monomials in x's. Newton's formulae are best used to prove the latter. To show that this action covers the action of $\mathcal{A}^{\mathrm{ev}}$ on $H^*(BU; Z_p)$, we recall the standard calculation of this action via the maps

$$H^*(BU; Z_p) \to H^*(BU(n); Z_p) \to H^*\big(BU(1)^{\times n}; Z_p\big) \cong H^*(BU(1); Z_p)^{\otimes n}$$

and identify the x's with the polynomial generators of $H^*(BU(1); Z_p)$, after reducing mod p. The action of $\mathcal{A}^{\mathrm{ev}}$ on $H^*(BU(1); Z_p)$ is calculated directly from the axioms. This completes the proof of (i).

Similarly, identify $H_*(BU; Z)$ with an algebra of symmetric functions by identifying a_j with $\sigma_i(y_1, y_2, \ldots, y_n, \ldots)$ in $Z[[y_1, y_2, \ldots, y_n, \ldots]]$. Let $\tilde{Q}^0(1) \equiv 1$, $\tilde{Q}^j(1) \equiv 0$ if $j > 1$, $\tilde{Q}^j(y_i) \equiv (-1)^{j+1} y_i^{1+j(p-1)}$ if $j > 0$, $\tilde{Q}^0(y_i) \equiv 0$. Extend to an action of $\tilde{R}$ on $Z[[y_1, y_2, \ldots, y_n]]$ by linearity and Cartan's formulae.

$$Q^j\left(y_1^i + y_2^i + \cdots + y_n^i + \cdots\right)$$
$$= (-1)^{i+j}\binom{j-1}{i-1}\left(y_1^{i+j(p-1)} + y_2^{i+j(p-1)} + \cdots + y_n^{i+j(p-1)} + \cdots\right)$$

is an easy consequence of Cartan's formula and the identity

$$\sum_{r=1}^{j-1}\binom{j-1-r}{i-2} = \binom{j-1}{i-1},$$

in turn an easy consequence of

$$\binom{j-2}{i-2} + \binom{j-2}{i-1} = \binom{j-1}{i-1}.$$

It follows immediately that the action of $\tilde{R}$ on $Z[[y_1, y_2, \ldots, y_n]]$ restricts to an action of $\tilde{R}$ on $H_*(BU; Z)$. The multiplicative and comultiplicative Cartan's formulae are readily verified.

To show that this action covers the action of R on $H_*(BU; Z_p)$, it suffices to show that the reduction of $\tilde{Q}^j(a_i)$, also denoted $\tilde{Q}^j(a_i)$, is equal to $Q^j(a_i)$, for all i and j. If $j = 0$ or $j = 1$ and $i = 1$, this is immediate from the definition of $Q^j(a_i)$ and the axioms for the Q's (see Kochman [1, p. 85]). Suppose $\tilde{Q}^{j'}(a_{i'}) = Q^{j'}(a_{i'})$ whenever $j' < j$ and $i' \leqslant i$ or $j' \leqslant j$ and $i' < i$. The comultiplicative Cartan formulas imply immediately that $\tilde{Q}^j(a_i) - Q^j(a_i)$ is primitive. In dimension $2i + 2j(p-1)$, the basic primitive $\mathcal{P}_{i+j(p-1)}$ is a polynomial in the a's with 1 as coefficient of $a_1^{i+j(p-1)}$. It suffices to show that the $a_1^{i+j(p-1)}$ coefficients of $\tilde{Q}^j(a_i)$ and $Q^j(a_i)$ are both zero. By Kochman [1, p. 88, Theorem 8], the $a_{i_1}a_{i_2}\ldots a_{i_k}$ coefficient of $Q^j(a_i)$ is zero if $i_l < i$, $l = 1, 2, \ldots, k$. By Kochman [1, p. 95, Lemma 23], the $c_{i_1}c_{i_2}\ldots c_{i_k}$ coefficient of $\mathcal{P}^j(c_i)$ is zero if $i_l < i$, $l = 1, 2, \ldots, k$. Since we have established that $\tilde{\mathcal{P}}^j(c_i)$ reduces to $\mathcal{P}^j(c_i)$, we may be content with proving

LEMMA 1. *The coefficient of $c_1^{i+j(p-1)}$ in $\tilde{\mathcal{P}}^j(c_i)$ is equal to that of $a_1^{j+i(p-1)}$ in $\tilde{Q}^i(c_j)$.*

The lemma will be proved as a corollary to Theorem 2 below. This completes the argument for Theorem 1.

REMARK. The two lemmas from Kochman are easy consequences of the Hopf algebra structures. The proof of Lemma 1 is quite elementary. However, the coefficients described in Lemma 1 are combinatorially defined, and it is by no means a trivial matter to prove combinatorially that they are zero when they should be. The obvious alternative to this argument is to show combina-

torially that the $\tilde{Q}^j$'s reduce to operations satisfying Nishida's relations on $H_*(BU)$, for Kochman's algorithm proceeds using a limited number of properties of the Q^j's, and all others are easily satisfied by the $\tilde{Q}^j$'s. This approach is, we believe, prohibitive.

2. For simplicity, we shall use the following notation. It is well known that a basis of primitives, $\{p_i\}_{i>0}$, of $H^*(BU; Z)$ (or Z_p) is the set of polynomials related to the Chern classes exactly as the symmetric powers are related to the elementary symmetric functions. Similarly, polynomials $\{\mathcal{P}_i\}_{i>0}$ of $H_*(BU; Z)$ (or Z_p) are a basis for the primitive submodule if they are related to the generators $\{a_i\}_{i>0}$ in the same manner. Let $c_{i,1} = (-1)^{i-1}p_i$ and $a_{i,1} = (-1)^{i-1}p_i$. (The significance of the double index will be made clear in [2].) Then Newton's formulae, expressing symmetric powers recursively, become

$$ic_i = \sum_{\substack{j+k=i \\ j=1,2,\ldots,i}} c_{j,1}c_k \quad \text{and} \quad ia_i = \sum_{\substack{j+k=i \\ j=1,2,\ldots,i}} a_{j,1}a_k$$

and Waring's formulae expressing symmetric powers nonrecursively become

$$c_{i,1} = \sum_{\substack{\Sigma\alpha j_\alpha = i \\ \Sigma j_\alpha = j}} (-1)^{j-1}\frac{i}{j}\binom{j}{j_1 j_2 \ldots j_i} c_1^{j_1}c_2^{j_2}\ldots c_i^{j_i}$$

and analogues for $a_{i,1}$.

Our plot is to express $\tilde{\mathcal{P}}^r(c_s)$ and $\tilde{Q}^r(a_s)$ as rational polynomials in the basic primitives, and simply replace the basic primitives by the Waring polynomials. The images of Newton formulae under the operations provide the basis for an inductive proof that the proposed rational polynomials in the basic primitives are correct. From several quarters, facts will be summoned to simplify coefficients once this procedure has been carried out.

From the definitions and multiplicative Cartan formulae, it is readily shown that $\tilde{\mathcal{P}}^r(c_s) = 0$ if $r > s$ and $\tilde{Q}^r(a_s) = 0$ if $r < s$, so we choose apt notation to write these "Newton-Cartan formulae":

$$\begin{aligned} j\tilde{\mathcal{P}}^{j-k}(c_j) &= \sum_{\substack{j'+j''=j \\ k'+k''=k}} \mathcal{P}^{j'-k'}(c_{j',1})\mathcal{P}^{j''-k''}(c_{j''}) \\ &= \sum_{\substack{j'+j''=j \\ k'+k''=k}} \binom{j'}{k'} c_{(j'-k')p+k',1}\mathcal{P}^{j''-k''}(c_{j''}) \end{aligned}$$

and

$$\begin{aligned} j\tilde{Q}^{j+k}(a_j) &= \sum_{\substack{j'+j''=j \\ k'+k''=k}} Q^{j'+k'}(c_{j',1})Q^{j''+k''}(c_{j''}) \\ &= \sum_{\substack{j'+j''=j \\ k'+k''=k}} (-1)^{k'}\binom{j'+k'-1}{j'-1} c_{(j'+k')p-k',1}Q^{j''+k''}(c_{j''}). \end{aligned}$$

The calculations of operations applied to primitives were carried out in thin disguise in the proof of Theorem 1. Some care must be taken with signs.

The following formulas are proved by induction on the pairs $(j, j-k)$ and $(j, j+k)$, with the lexicographic partial ordering, by substituting into the Newton-Cartan formulae:

THEOREM 2.

$$\tilde{\mathcal{P}}^{j-k}(c_j) = \sum \prod \frac{1}{l_\alpha!}\left\{\frac{1}{j_\alpha}\binom{j_\alpha}{k_\alpha} c_{(j_\alpha - k_\alpha)p + k_\alpha, 1}\right\}^{l_\alpha}$$

admissible summands: $\Sigma_\alpha k_\alpha l_\alpha = k$, $\Sigma_\alpha j_\alpha l_\alpha = j$, $j_\alpha \neq j_\beta$ *or* $k_\alpha \neq k_\beta$ *if* $\alpha \neq \beta$.

$$\tilde{Q}^{j+k}(a_j) = \sum \frac{1}{l_\alpha!}\left\{\frac{(-1)^{k_\alpha}}{j_\alpha}\binom{j_\alpha + k_\alpha - 1}{j_\alpha - 1} a_{(j_\alpha + k_\alpha)p - k_\alpha, 1}\right\}^{l_\alpha}$$

$$= (-1)^k \sum \frac{1}{l_\alpha!}\left\{\frac{1}{j_\alpha + k_\alpha}\binom{j_\alpha + k_\alpha}{j_\alpha} a_{(j_\alpha + k_\alpha)p - k_\alpha, 1}\right\}^{l_\alpha}$$

admissible summands: $\Sigma_\alpha k_\alpha l_\alpha = k$, $\Sigma_\alpha j_\alpha l_\alpha = j$, $j_\alpha \neq j_\beta$ *or* $k_\alpha \neq k_\beta$ *if* $\alpha \neq \beta$.

For the "Steenrod powers", the induction begins for each j with $k - j = 0$. Since $\tilde{\mathcal{P}}^0(c_j) = c_j$, the calculation in this case amounts to representing the jth elementary symmetric function as a rational polynomial in the symmetric powers. The proof is quite easy, using Newton's formulae. For the "Dyer-Lashof operations", the induction begins at an awkward moment for each j, with $j + k = j$. However, a quick inspection of the formulas indicates that $\tilde{Q}^j(a_j) = T_j(a_{1,1}, a_{2,1}, \ldots, a_{j,1})$ if and only if $\tilde{\mathcal{P}}^j(c_j) = T_j(c_{1,1}, c_{2,1}, \ldots, c_{j,1})$. By the definitions of $\mathcal{P}$'s and Q's, it is easily established that, in the notation of Theorem 1,

$$\tilde{\mathcal{P}}^j(c_j) \text{“}=\text{”} \sum_{\text{symm}} x_1^p x_2^p \ldots x_j^p \quad \text{and} \quad \tilde{Q}^j(a_j) \text{“}=\text{”} \sum_{\text{symm}} y_1^p y_2^p \ldots y_j^p.$$

Hence, if the proof of the cohomology formulas is completed first, the first inductive steps for the homology formulas are available all at once.

COROLLARY 1. (i) *The coefficient of* $c_1^{(j-k)p+k}$ *in* $\tilde{\mathcal{P}}^{j-k}(c_j)$ *is*

$$\sum \prod \frac{(-1)^{l_\alpha}}{l_\alpha!}\left\{\frac{(-1)^{j_\alpha}}{j_\alpha}\binom{j_\alpha}{k_\alpha}\right\}^{l_\alpha}.$$

(ii) *The coefficient of* $a_1^{(j+k)p-k}$ *in* $\tilde{Q}^{j+k}(c_j)$ *is*

$$\sum \frac{(-1)^{l_\alpha}}{l_\alpha!}\left\{\frac{(-1)^{j_\alpha + k_\alpha}}{j_\alpha + k_\alpha}\binom{j_\alpha + k_\alpha}{j_\alpha}\right\}^{l_\alpha}$$

(*admissibility is defined as in Theorem* 2).

PROOF. From Waring's formulas, the coefficient of $c_1^{(j_\alpha - k_\alpha)p+k}$ in $c_{(j_\alpha - k_\alpha)p+k_\alpha,1}$ is $(-1)^{(j_\alpha - k_\alpha)p+k_\alpha-1}$ and that of $a_1^{(j_\alpha+k_\alpha)p-k_\alpha}$ in $a_{(j_\alpha+k_\alpha)p-k_\alpha,1}$ is $(-1)^{(j_\alpha+k_\alpha)p-k_\alpha-1}$.

COROLLARY 2. *Lemma* 1.

PROOF. A straightforward reparametrization.

THEOREM 3.

$$\tilde{\mathcal{P}}^{j-k}(c_j) = \sum (-1)^{\Sigma r_\alpha} \prod_\alpha \frac{(-1)^{l_\alpha}}{l_\alpha!}$$
$$\times \left\{ \frac{(j_\alpha - k_\alpha)p + k_\alpha}{j_\alpha} \binom{j_\alpha}{k_\alpha} \frac{1}{t_\alpha} \binom{t_\alpha}{t_{\alpha 1} t_{\alpha 2} \cdots t_{\alpha(j_\alpha - k_\alpha)p+k_\alpha}} \right\}^{l_\alpha}$$
$$\times c_1^{r_1} c_2^{r_2} \cdots c_{(j-k)p+k}^{r_{(j-k)p+k}},$$

admissible summands: $\Sigma_\alpha k_\alpha l_\alpha = k$, $\Sigma_\alpha j_\alpha l_\alpha = j$, $\Sigma_\gamma \gamma t_{\alpha\gamma} = (j_\alpha - k_\alpha)p + k_\alpha$, $\Sigma_\alpha l_\alpha t_{\alpha\gamma} = r_\gamma$, $\Sigma_\gamma t_{\alpha\gamma} = t_\alpha$.

$$\tilde{Q}^{j+k}(c_j) = (-1)^k \Sigma (-1)^{\Sigma r\gamma} \prod \frac{(-1)^{l_\alpha}}{l_\alpha!}$$
$$\times \left\{ \frac{(j_\alpha + k_\alpha)p - k_\alpha}{j_\alpha + k_\alpha} \binom{j_\alpha + k_\alpha}{j_\alpha} \frac{1}{t_\alpha} \binom{t_\alpha}{t_{\alpha 1} t_{\alpha 2} \cdots t_{\alpha(j_\alpha + k_\alpha)p-k_\alpha}} \right\}^{l_\alpha}$$
$$\times a_1^{r_1} a_2^{r_2} \cdots a_{(j+k)p-k}^{r_{(j+k)p-k}},$$

admissible summands: $\Sigma_\alpha k_\alpha l_\alpha = k$, $\Sigma_\alpha j_\alpha l_\alpha = j$, $\Sigma_\gamma \gamma t_{\alpha\gamma} = (j_\alpha + k_\alpha)p - k_\alpha$, $\Sigma_\alpha l_\alpha t_{\alpha\gamma} = r_\gamma$, $\Sigma_\gamma t_{\alpha\gamma} = t_\alpha$.

PROOF. Replace $c_{(j_\alpha - k_\alpha)p+k_\alpha,1}$ and $a_{(j_\alpha+k_\alpha)p-k_\alpha,1}$ by the Waring polynomials in the c's and a's and apply multinomial expansion. It is important to note that the restrictions imposed in Theorem 2, $(j_\alpha, k_\alpha) \neq (j_\beta, k_\beta)$ if $\alpha \neq \beta$, no longer apply because of this multinomial expansion.

3. We shall be able to restrict the admissible summands indicated in Theorem 3 as follows: (i) $\tilde{\mathcal{P}}^0(c_j) = c_j$, so that, reading from Theorem 3, with $k = j$, we see that

$$\sum \prod \frac{(-1)^{l_\alpha}}{l_\alpha!} \left\{ \frac{1}{t_\alpha} \binom{t_\alpha}{t_{\alpha 1} t_{\alpha 2} \cdots t_{\alpha j_\alpha}} \right\}^{l_\alpha} = 0$$

(admissible summands $\Sigma_\alpha j_\alpha l_\alpha = j$, $\Sigma_\alpha l_\alpha t_{\alpha,\gamma} = r\gamma$, $\Sigma_\gamma t_{\alpha\gamma} = t_\alpha$) unless $l_\alpha = 1$, $t_\alpha = 1, j_\alpha = j$, and $t_{\alpha\gamma} = 0$ unless $\gamma = j$.

It follows that nothing is lost by restricting the admissible summands of Theorem 3 for $\tilde{\mathcal{P}}^{j-k}(c_j)$ by requiring that at most one $j_\alpha = k_\alpha$, and then $t_\alpha = 1$, $l_\alpha = 1$, and for $\tilde{Q}^{j+k}(a_j)$ by requiring that at most one $k_\alpha = 0$ and then $t_\alpha = 1$, $l_\alpha = 1$.

(ii) By Kochman [1, Theorem 8], the product filtration degree of $Q^r(a_n)$ is $\leqslant (r-n)(p-1)+p$ and the only monomial of degree $(r-n)(p-1)+p$ with nonzero coefficient is $a_n^p a^{(r-n)(p-1)}$. By Kochman [1, Lemma 23], the product filtration degree of $\mathcal{P}^r(c_n)$ is $\leqslant p$. Thus, after reducing mod p, nothing is lost by restricting $\Sigma_\gamma r_\gamma \leqslant p$ in $\mathcal{P}^{j-k}(c_j)$ and $\Sigma_\gamma r_\gamma \leqslant k(p-1)+p$ in $Q^{j+k}(a_j)$.

(iii) $\tilde{\mathcal{P}}^j(c_j)$ reduces to c_j^p and $\tilde{Q}^j(a_j)$ reduces to a_j^p, so that nothing is lost by imposing the restrictions that $k_\alpha = 0$ implies $t_\alpha = t_{\alpha j_\alpha} = p$ and $l_\alpha = 1$ in the formulas for $\mathcal{P}^{j-k}(c_j)$ and $Q^{j+k}(a_j)$. $\Sigma_\alpha l_\alpha t_\alpha = \Sigma_{\alpha,\gamma} l_\alpha t_{\alpha,\gamma} = \Sigma_\gamma r_\gamma \leqslant p$ and $t_\alpha = p$ if $k_\alpha = 0$ clearly implies $k = 0$ if any $k_\alpha = 0$.

THEOREM 4.

$$\mathcal{P}^{j-k}(c_j) = \sum (-1)^{\Sigma r_\gamma} \prod \frac{(-1)^{l_\alpha}}{l_\alpha!}$$

(i)
$$\times \left\{ \frac{(j_\alpha - k_\alpha)p + k_\alpha}{j_\alpha} \binom{j_\alpha}{k_\alpha} \frac{1}{t_\alpha} \binom{t_\alpha}{t_{\alpha 1} \ldots t_{\alpha(j_\alpha - k_\alpha)p + k_\alpha}} \right\}^{l_\alpha}$$
$$\times c_1^{r_1} c_2^{r_2} \ldots c_{(j-k)p+k}^{r_{(j-k)p+k}}$$

admissible summands: $\Sigma_\alpha k_\alpha l_\alpha = k$, $\Sigma_\alpha j_\alpha l_\alpha = j$, $\Sigma_\gamma \gamma t_{\alpha\gamma} = (j_\alpha - k_\alpha)p + k_\alpha$, $\Sigma_\alpha l_\alpha t_{\alpha\gamma} = r_\gamma$, $\Sigma_\gamma t_{\alpha\gamma} = t_\alpha$, $\Sigma_\gamma r_i \leqslant p$, $j_\alpha > k_\alpha$ *with at most one exception, for which* $t_\alpha = l_\alpha = 1$, $k_\alpha > 0$ *unless* $k = 0$.

$$Q^{j+k}(a_j) = (-1)^k \sum (-1)^{\Sigma r_\gamma} \prod \frac{(-1)^{l_\alpha}}{l_\alpha!}$$

(ii)
$$\times \left\{ \frac{(j_\alpha + k_\alpha)p - k_\alpha}{j_\alpha + k_\alpha} \binom{j_\alpha + k_\alpha}{j_\alpha} \frac{1}{t_\alpha} \binom{t_\alpha}{t_{\alpha 1} \ldots t_{\alpha(j_\alpha + k_\alpha)p - k_\alpha}} \right\}^{l_\alpha}$$
$$\times a_1^{r_1} a_2^{r_2} \ldots a_{(j+k)p-k}^{r_{(j+k)p-k}},$$

admissible summands: $\Sigma_\alpha k_\alpha l_\alpha = k$, $\Sigma_\alpha j_\alpha l_\alpha = j$, $\Sigma_\gamma \gamma t_{\alpha\gamma} = (j_\alpha + k_\alpha)p - k_\alpha$, $\Sigma_\alpha l_\alpha t_{\alpha\gamma} = r_\gamma$, $\Sigma_\gamma t_{\alpha\gamma} = t_\alpha$, $\Sigma_\gamma r_\gamma \leqslant p + k(p-1)$, $k_\alpha > 0$ *unless* $t_\alpha = t_{\alpha j_\alpha}$ *and* $l_\alpha = 1, j_\alpha > 0$.

Note. If all l_α and t_α are less than p,

$$\frac{(j_\alpha - k_\alpha)p + k_\alpha}{j_\alpha} \binom{j_\alpha}{k_\alpha}$$

can be simplified to

$$\binom{j_\alpha - 1}{k_\alpha - 1}$$

and

$$\frac{(j_\alpha + k_\alpha)p - k_\alpha}{j_\alpha + k_\alpha}\binom{j_\alpha + k_\alpha}{j_\alpha}$$

to

$$-\binom{j_\alpha + k_\alpha - 1}{k_\alpha - 1}.$$

We point out the following special cases:

(i)
$$\mathcal{P}^1(c_j) = \sum_{\substack{\Sigma\gamma r_\gamma = j+p-1 \\ \Sigma r_\gamma < p}} (-1)^{\Sigma_\gamma r_\gamma - 1} \frac{1}{\sum_\gamma r_\gamma}\binom{\sum_\gamma r_\gamma}{r_1 r_2 \ldots r_{(j-1)p+1}}$$
$$\times \left\{ j + (p-1) - \sum_{u=1}^{j-1} \frac{j + (p-1) - u}{\sum_\gamma r_\gamma - 1} \cdot r_u \right\} c_1^{r_1} c_2^{r_2} \ldots c_{j+p-1}^{r_{j+p-1}}.$$

If $j > (p-1)$ this can further be simplified by replacing the bracketed expression by

$$\left\{ \sum_{u=0}^{p-2} \frac{(p-1) - u}{\sum_\gamma r_\gamma - 1} r_{u+j} \right\} \quad \text{if } c_1^{r_1} c_2^{r_2} \ldots c_{j+(p-1)}^{r_{j+p-1}} \neq c_{j+(p-1)}.$$

(ii)
$$\mathcal{P}^{j-1}(c_j) = \sum_{\substack{\Sigma\gamma r_\gamma = (j-1)p+1 \\ \Sigma r_\gamma < p}} (-1)^{\Sigma_\gamma r_\gamma - 1} \frac{1}{\sum_\gamma r_\gamma}\binom{\sum_\gamma r_\gamma}{r_1 r_2 \ldots r_{(j-1)p+1}}$$
$$\times c_1^{r_1} c_2^{r_2} \ldots c_{(j-1)p+1}^{r_{(j-1)p+1}}.$$

(iii) The coefficient of $c_{35}^2 c_{13}^3 c_{10}$ in $\mathcal{P}^{14}(c_{35})$ is 3 (mod 7).

(iv)
$$Q^{j+1}(a_j) = \sum_{\substack{\Sigma_\gamma r_\gamma = r \\ \Sigma_\gamma n_\gamma = n \\ \Sigma\gamma r_\gamma = (j+1)p-1}} (-1)^{r+n} \frac{(j-n+1)p - 1}{r - np}$$
$$\times \binom{r - np}{r_1 - n_1 p \; r_2 - n_2 p \ldots r_\gamma - n_\gamma p \ldots}$$
$$\times a_1^{r_1} a_2^{r_2} \ldots a_{(j+1)p-1}^{r_{(j+1)p-1}}.$$

4. If $p = 2$, superscripts on operations should be doubled for calculations in BU. If superscripts are undoubled, with the obvious substitutions, the formulas portray the actions of the algebras in homology and cohomology of BO. (Wu's formulas are, on the face of it, simpler, but can easily be derived.) In similar fashion, elementary adjustments can be made for BSp, and at least in cohomology, BSU. The homology of BSU will be treated elsewhere; see, however, Shay [4] and Kochman [1, Theorem 20], for partial information.

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