PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 63, Number 2, April 1977

MAXIMAL LOGICS

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ABSTRACT. In this paper we present a general method for producing logics on various classes of models which are maximal with respect to a Łoś ultraproducts theorem. As a corollary we show that $\mathfrak{L}^{\mathrm{Top}}$ is maximal. We also show that these maximal logics satisfy the Souslin-Kleene property.

0. **Introduction.** In this paper we will prove that there is a strongest logic for certain classes of models with a Łoś ultraproduct theorem.

The motivation comes from two sources. The first is the area of abstract logic and model theory. P. Lindström first proved that $\mathfrak{L}_{\omega\omega}$ is the strongest logic which satisfies the compactness and Löwenheim-Skolem theorem. K. J. Barwise **[B-1]** expanded, simplified, and strengthened these results by formulating abstract model theory in a category-theoretic framework.

The second area is topological model theory. In **[S-1]** we presented a topological logic using generalized quantifiers. This logic is formed by adding a quantifier symbol Qx to $\mathfrak{L}_{\omega\omega}$, denoted by $\mathfrak{L}(Q)$, where the interpretation of $Qx\phi(x)$ is that the set defined by $\phi(x)$ is "open". Another logic, denoted by $\mathfrak{L}(Q^n)_{n\in\omega}$, is formed by adding $Q^n x_1, \ldots, x_n$ for each n so that the interpretation of $Q^n x_1, \ldots, x_n \phi(x_1, \ldots, x_n)$ is that the set defined by $\phi(x_1, \ldots, x_n)$ is "open in the nth product topology". However, they are not the strongest logics even though they both satisfy the compactness and Löwenheim-Skolem theorems.

More recently, S. Garavaglia and T. McKee (see **[G-1]** or **[McK]**) have found an extension, $\mathfrak{L}^{\mathrm{Top}}$, of $\mathfrak{L}(Q)$ and $\mathfrak{L}(Q^n)_{n\in\omega}$ which has many desirable properties, e.g. compactness, Löwenheim-Skolem, interpolation, and an isomorphic ultrapowers theorem.

Hence, one is naturally led to the question of when there is a strongest logic with first order properties. In this paper we give a construction of the strongest logic with a Łoś ultraproduct theorem. We then show that these maximal logics have the Souslin-Kleene property.

1. **Abstract logics.** We will assume that the reader is familiar with the basic notions of first order model theory (e.g. many-sorted logics, ultrafilters, and ultraproducts), topology and category theory.

Received by the editors December 10, 1975 and, in revised form, July 15, 1976.
AMS (MOS) subject classifications (1970). Primary 02H99; Secondary 02C99, 02G99, 18B99.
[1] This research was supported by NSF grant MPS-74-08550.

Take a first order model $\mathfrak{A}$, $q \subseteq \mathcal{P}(A)$, and form $(\mathfrak{A}, q)$. $(\mathfrak{A}, q)$ is called a *weak model.* (If q is a topology then $(\mathfrak{A}, q)$ is called *topological.*) If we take a class of weak models, Mod, we can give a definition of a logic on Mod.

Our notion of a logic is similar to that of Barwise [**B-1**] and we will assume some familiarity with it. We take the set of objects to be $\mathfrak{L}$, the class of languages. The morphisms will be the k-morphisms in [**B-1**].

We define a logic, $\mathfrak{L}^*$, to consist of a syntax and a semantics. The syntax is a functor $*$ on $\mathfrak{L}$. The elements $\mathfrak{L}^*$ for $L \in \mathfrak{L}$ are called $\mathfrak{L}^*$ sentences. The functor $*$ satisfies the following axiom:

OCCURRENCE AXIOM. For every $\mathfrak{L}^*$ sentence ϕ there is a smallest (under $\subseteq$) language L_ϕ in $\mathfrak{L}$ such that $\phi \in L_\phi^*$.

The semantics of $\mathfrak{L}^*$ is a relation $\models^{\mathfrak{L}^*}$ such that if $(\mathfrak{A}, q) \models^{\mathfrak{L}^*} \phi$, then $\mathfrak{A}$ is an L-structure for some L in $\mathfrak{L}$ and $\phi \in L^*$. It satisfies the following axiom:

ISOMORPHISM AXIOM. If $(\mathfrak{A}, q) \models^{\mathfrak{L}^*} \phi$ and $(\mathfrak{A}, q) \cong (\mathfrak{B}, r)$ (i.e. $(\mathfrak{A}, q)$, $(\mathfrak{B}, r)$ are isomorphic as 2-sorted structures), then $(\mathfrak{B}, r) \models^{\mathfrak{L}^*} \phi$.

A logic on Mod which has important applications is the 2-sorted logic, $\mathfrak{L}_2^{\mathrm{Mod}}$. (We will write $\mathfrak{L}_2$ for $\mathfrak{L}_2^{\mathrm{Mod}}$ when the meaning is understood.) For $L \in \mathfrak{L}$, L^2 is the set of 2-sorted sentences (i.e. built up from the constants, predicates, functions, individual variables, set variables, equality and $\in$ using $\vee$, $\neg$, $\exists x$, $\exists X$). Note that $\in$ has its standard meaning and is a logical symbol. If $(\mathfrak{A}, q) \in$ Mod then

$$(\mathfrak{A}, q) \models^{\mathfrak{L}_2} \phi$$

will be the usual satisfaction relation.

We say that ϕ in $\mathfrak{L}^*$ is $EC_{\mathfrak{L}^\#}(L)$ if and only if there is a $\psi \in L^\#$ such that

$$\mathrm{Mod}_L^{\mathfrak{L}^*}(\phi) = \mathrm{Mod}_L^{\mathfrak{L}^\#}(\psi)$$

where

$$\mathrm{Mod}_L^{\mathfrak{L}}(\phi) = \{(\mathfrak{A}, q) | (\mathfrak{A}, q) \models^{\mathfrak{L}} \phi, (\mathfrak{A}, q) \text{ an } L\text{-structure}\}.$$

Suppose we have two logics, $\mathfrak{L}^*$, $\mathfrak{L}^\#$, on a class of models, Mod. Then we can define an ordering between them which is a measure of their strength of expressibility. We say that $\mathfrak{L}^\#$ is as strong as $\mathfrak{L}^*$, $\mathfrak{L}^\# \geqslant \mathfrak{L}^*$, if for every $\mathfrak{L}^*$-sentence ϕ there is an $\mathfrak{L}^\#$-sentence ψ such that

(i) Every symbol occurring in ψ occurs in ϕ, i.e. $L_\psi \subseteq L_\phi$.
(ii) $\mathrm{Mod}^{\mathfrak{L}^*}(\phi) = \mathrm{Mod}^{\mathfrak{L}^\#}(\psi)$.

Again taking an arbitrary class of models, Mod, let $T = \{q | (\mathfrak{A}, q) \in \mathrm{Mod}$ for some $\mathfrak{A}\}$ and suppose we are given an $\mathfrak{F}$: $T \to T$, a map, so that $\mathfrak{F}(q) = \mathfrak{F}(\mathfrak{F}(q))$ and for each q, if $(\mathfrak{A}, q) \in$ Mod then $(\mathfrak{A}, \mathfrak{F}(q)) \in$ Mod . We can define two logics based on this $\mathfrak{F}$. Let ϕ be an $\mathfrak{L}_2$-sentence; then ϕ is called *$\mathfrak{F}$-invariant* if and only if for all $(\mathfrak{A}, q) \in$ Mod,

$$(\mathfrak{A}, q) \models^{\mathfrak{L}_2} \phi \text{ if and only if } (\mathfrak{A}, \mathfrak{F}(q)) \models^{\mathfrak{L}_2} \phi.$$

We define $\mathfrak{L}_2^{\mathfrak{F}}$ to be the sublogic of $\mathfrak{L}_2$ (on Mod) which consists of the $\mathfrak{F}$-invariant sentences. Taking $\mathrm{Mod}(\mathfrak{F}) = \{(\mathfrak{A}, \mathfrak{F}(q)) | (\mathfrak{A}, q) \in \mathrm{Mod}\}$, we can define a logic, $\mathfrak{L}^{\mathfrak{F}}$, on $\mathrm{Mod}(\mathfrak{F})$ as follows:

(a) the $L^{\mathfrak{F}}$-sentences are just the $\mathfrak{F}$-*invariant* ones,

(b) $(\mathfrak{A}, \mathfrak{F}(q)) \models^{\mathfrak{L}^{\mathfrak{F}}} \phi$ if and only if $(\mathfrak{A}, q) \models^{\mathfrak{L}_2} \phi$.

If we are given a logic $\mathfrak{L}^*$ on $\mathrm{Mod}(\mathfrak{F})$ where $\mathfrak{F}$ is as above then we can define an $\mathfrak{L}_2^*$ on Mod. $\mathfrak{L}_2^*$ will have the same sentences as $\mathfrak{L}^*$ but the satisfaction relation will be defined as follows: if $(\mathfrak{A}, q) \in \mathrm{Mod}$ then

$$(\mathfrak{A}, q) \models^{\mathfrak{L}_2^*} \phi \text{ if and only if } (\mathfrak{A}, \mathfrak{F}(q)) \models^{\mathfrak{L}^*} \phi.$$

This is the analogue of $\models^{\mathfrak{L}^{\mathfrak{F}}}$ for $\mathfrak{L}^*$. We can then prove the following:

LEMMA 1. *If* $\mathfrak{L}_2^* \leqslant \mathfrak{L}_2$ *then* $\mathfrak{L}^* \leqslant \mathfrak{L}^{\mathfrak{F}}$.

PROOF. Assume $\mathfrak{L}_2^* \leqslant \mathfrak{L}_2$ and suppose that $\phi \in L^*$. Since ϕ is $EC_{\mathfrak{L}_2}(L)$ we have $\psi \in L^2$ such that $\mathrm{Mod}^{\mathfrak{L}_2^*}(\phi) = \mathrm{Mod}^{\mathfrak{L}_2}(\psi)$ but then ψ is $\mathfrak{F}$-invariant so $\mathrm{Mod}^{\mathfrak{L}^*}(\phi) = \mathrm{Mod}^{\mathfrak{L}^{\mathfrak{F}}}(\psi)$.

Suppose we are given a class of models, Mod, such that if $(\mathfrak{A}_\gamma, q_\gamma) \in \mathrm{Mod}$, $\gamma < \lambda$ and U is an ultrafilter on λ, then we have that $\prod_U (\mathfrak{A}_\gamma, q_\gamma) = (\prod_U \mathfrak{A}_\gamma, \prod_U q_\gamma)$ is in Mod (i.e. Mod is closed under ultraproducts, where $\prod_U \mathfrak{A}_\gamma$ is the usual ultraproduct on the $\mathfrak{A}_\gamma$, $\gamma < \lambda$, and $\prod_U q_\gamma = \{\prod_U [\mathcal{O}_\gamma] \| \mathcal{O}_\gamma \in q_\gamma\}$, where $\prod_U [\mathcal{O}_\gamma] = \{[f]_U | \{\gamma | f(\gamma) \in \mathcal{O}_\gamma\} \in U\}$). Furthermore if $\mathfrak{F}(\prod_U q_\gamma) = \mathfrak{F}(\prod_U \mathfrak{F}(q_\gamma))$, then we can give a notion of ultraproduct for $\mathfrak{L}^*$ on $\mathrm{Mod}(\mathfrak{F})$ as follows:

$$\prod_U^{\mathfrak{F}} (\mathfrak{A}_\gamma, q_\gamma) = \left(\prod_U \mathfrak{A}_\gamma, \mathfrak{F}\left(\prod_U q_\gamma\right)\right) \in \mathrm{Mod}(\mathfrak{F}).$$

This naturally leads to the question of whether there is an analogue to the Łoś ultraproduct theorem for $\mathfrak{L}^*$ (i.e. $\prod_U^{\mathfrak{F}} (\mathfrak{A}_\gamma, q_\gamma) \models^{\mathfrak{L}^*} \phi$ if and only if $\{\gamma | (\mathfrak{A}_\gamma, \mathfrak{F}(q_\gamma)) \models^{\mathfrak{L}^*} \phi\} \in U$ for all ϕ in $\mathfrak{L}^*$). The following lemma clarifies the situation.

LEMMA 2. $\mathfrak{L}_2^*$ *has a Łoś theorem* (*on* Mod) *if and only if* $\mathfrak{L}^*$ *has a Łoś theorem* (*on* $\mathrm{Mod}(\mathfrak{F})$).

PROOF. (IF) Assume $\mathfrak{L}^*$ is such that for all ϕ in $\mathfrak{L}^*$, $\prod_U^{\mathfrak{F}} (\mathfrak{A}_\gamma, q_\gamma) \models^{\mathfrak{L}^*} \phi$ if and only if $\{\gamma | (\mathfrak{A}_\gamma, \mathfrak{F}(q_\gamma)) \models^{\mathfrak{L}^*} \phi\} \in U$. By the definition of $\mathfrak{L}_2^*$,

$$\begin{aligned}\prod_U (\mathfrak{A}_\gamma, q_\gamma) = \left(\prod_U \mathfrak{A}_\gamma, \prod_U q_\gamma\right) \models^{\mathfrak{L}_2^*} \phi \text{ iff } \left(\prod_U \mathfrak{A}_\gamma, \mathfrak{F}\left(\prod_U q_\gamma\right)\right) \models^{\mathfrak{L}^*} \phi \\ \text{iff } \left(\prod_U \mathfrak{A}_\gamma, \mathfrak{F}\left(\prod_U \mathfrak{F}(q_\gamma)\right)\right) \models^{\mathfrak{L}^*} \phi \text{ iff } \{\gamma | (\mathfrak{A}_\gamma \mathfrak{F}(q_\gamma)) \models^{\mathfrak{L}^*} \phi\} \in U \\ \text{iff } \{\gamma | (\mathfrak{A}_\gamma, q_\gamma) \models^{\mathfrak{L}_2^*} \phi\} \in U.\end{aligned}$$

(ONLY IF) The proof is similar to the if direction.

EXAMPLES. (a) The most interesting example is topology. Let $\mathrm{Mod}_b = \{(\mathfrak{A}, q) | q$ is a base for a topology on $A\}$ and $\mathrm{Top}(q)$ be the topology generated by q. Then

$$\mathrm{Top}(\mathrm{Top}(q)) = \mathrm{Top}(q), \quad \mathrm{Top}\Big(\prod_U q_\gamma\Big) = \mathrm{Top}\Big(\prod_U \mathrm{Top}(q_\gamma)\Big)$$

and $\mathfrak{L}^{\mathrm{Top}}$ has a Łoś theorem.

(b) Let $\mathrm{Mod}_{\mathrm{Fil}} = \{(\mathfrak{A}, q) | q$ is a base for a filter on $A\}$; then $\mathrm{Fil}(q) =$ the filter on A generated by q. Hence $\mathrm{Fil}(q) = \mathrm{Fil}(\mathrm{Fil}(q))$, $\mathrm{Fil}(\prod_U q_\gamma) = \mathrm{Fil}(\prod_U \mathrm{Fil}(q_\gamma))$ and $\mathfrak{L}^{\mathrm{Fil}}$ satisfies a Łoś theorem.

(c) Let $\mathrm{Mod}_{\mathrm{BA}} = \{(\mathfrak{A}, q) | q \subseteq \mathcal{P}(A)\}$. Then $\mathrm{BA}(q)$ $(\mathrm{CBA}(q))$ is the (complete) Boolean algebra generated by q and $\mathrm{BA}(\prod_U q_\gamma) = \mathrm{BA}(\prod_U \mathrm{BA}(q_\gamma))$ $(\mathrm{CBA}(\prod_U q_\gamma) = \mathrm{CBA}(\prod_U \mathrm{CBA}(q_\gamma)))$. $\mathfrak{L}^{\mathrm{BA}}$ and $\mathfrak{L}^{\mathrm{CBA}}$ have Łoś theorems.

If $\mathfrak{L}^*$ has a Łoś theorem, we say that $\mathfrak{L}^*$ has the *Łoś property.*

The last result we need is a two-sorted version of Shelah's isomorphic ultrapowers theorem.

THEOREM. *If $(\mathfrak{A}, q) \equiv_{\mathfrak{L}_2} (\mathfrak{B}, r)$, then there is an ultrafilter U on a cardinal κ such that $\prod_U (\mathfrak{A}, q) \cong \prod_U (\mathfrak{B}, r)$.*

PROOF. This two-sorted version is analogous to the proof found in **[C-N]**.

Now we can prove

THEOREM. *Let $\mathfrak{L}^*$ be a logic on* $\mathrm{Mod}(\mathcal{F})$. *If $\mathfrak{L}^*$ has the Łoś property, then $\mathfrak{L}^* \leqslant \mathfrak{L}^{\mathcal{F}}$.*

(One should note that we have not placed any restrictions on the size of L.)

PROOF. Assume $\mathfrak{L}^*$ has the Łoś property. Assume furthermore that $\mathfrak{L}^* \nleqslant \mathfrak{L}^{\mathcal{F}}$ and we will proceed to produce a contradiction.

We know that $\mathfrak{L}_2^* \nleqslant \mathfrak{L}_2$ by Lemma 1 and the assumption. Let $\phi \in L^*$ be a sentence such that ϕ is not $EC_{\mathfrak{L}_2}(L)$. Since L is a set we have that $|L| \leqslant \lambda$, λ some infinite cardinal.

Take U to be a λ-regular ultrafilter on λ which exists. By the definition of regular, we know that there is a set $\mathfrak{X} \subseteq U$ of power λ such that each $\gamma \in \lambda$ belongs to only finitely many $X \in \mathfrak{X}$.

Let $\{\psi_\gamma\}_{\gamma \in \lambda}$ and $\{X_\delta\}_{\delta < \lambda}$ be enumerations of L^2 and $\mathfrak{X}$, respectively. Then we claim that for each $\delta < \lambda$ there is $(\mathfrak{A}_\delta, q_\delta)$, $(\mathfrak{B}_\delta, r_\delta) \in \mathrm{Mod}$ such that for each $\gamma \in \Sigma(\delta)$,

$$(*) \qquad \begin{gathered} (\mathfrak{A}_\delta, q_\delta) \models^{\mathfrak{L}_2} \psi_\gamma \text{ iff } (\mathfrak{B}_\delta, r_\delta) \models^{\mathfrak{L}_2} \psi_\gamma \quad \text{and} \\ (\mathfrak{A}_\delta, q_\delta) \models^{\mathfrak{L}_2^*} \phi, \ (\mathfrak{B}_\delta, r_\delta) \not\models^{\mathfrak{L}_2^*} \phi, \end{gathered}$$

where $\Sigma(\delta) = \{\gamma | \delta \in X_\gamma\}$ which is finite by the selection of $\mathfrak{X}$. Assume $(*)$ does not hold and that $\mathrm{Mod}(\theta)^0 = \mathrm{Mod}(\theta)$ and $\mathrm{Mod}(\theta)^1 = \mathrm{Mod}(\theta)^c$. By our assumption we know that for each $\eta\colon \Sigma(\delta) \to \{0, 1\}$ either

$$\bigcap_{\gamma \in \Sigma(\delta)} \operatorname{Mod}(\psi_\gamma)^{\eta(\gamma)} \cap \operatorname{Mod}(\phi) = \varnothing \quad \text{or}$$

$$\bigcap_{\gamma \in \Sigma(\delta)} \operatorname{Mod}(\psi_\gamma)^{\eta(\gamma)} \cap \operatorname{Mod}(\phi)^c = \varnothing.$$

Then since $\bigcup_\eta \bigcap_{\gamma \in \Sigma(\delta)} \operatorname{Mod}(\psi_\gamma)^{\eta(\gamma)} = \operatorname{Mod}$ and $\Sigma(\delta)$ is finite, ϕ would be $EC_{\mathfrak{L}_2}(L)$. This is a contradiction.

Since $\mathfrak{L}^*$ has the Łoś property, Lemma 2 implies that $\mathfrak{L}_2^*$ has the Łoś property, so

$$\prod_U (\mathfrak{A}_\delta, q_\delta) \models^{\mathfrak{L}_2^*} \phi \quad \text{and} \quad \prod_U (\mathfrak{B}_\delta, r_\delta) \not\models^{\mathfrak{L}_2^*} \phi.$$

Also

$$\prod_U (\mathfrak{A}_\delta, q_\delta) \equiv_{\mathfrak{L}_2} \prod_U (\mathfrak{B}_\delta, r_\delta).$$

This follows from the following observation. If $\psi \in L^2$ then $\psi = \psi_\alpha$ for some $\alpha < \lambda$. We know from $(*)$ that we have for each $\delta \in X_\alpha \in U$,

$$(\mathfrak{A}_\delta, q_\delta) \models^{\mathfrak{L}_2} \psi \text{ iff } (\mathfrak{B}_\delta, r_\delta) \models^{\mathfrak{L}_2} \psi.$$

This yields the result.

To finish the proof we use Shelah's isomorphic ultrapowers theorem to obtain an ultrafilter, V, such that $\prod_V \prod_U (\mathfrak{A}_\delta, q_\delta) \cong \prod_V \prod_U (\mathfrak{B}_\delta, r_\delta)$,

$$\prod_V \prod_U (\mathfrak{A}_\delta, q_\delta) \models^{\mathfrak{L}_2^*} \phi \quad \text{and} \quad \prod_V \prod_U (\mathfrak{B}_\delta, r_\delta) \not\models^{\mathfrak{L}_2^*} \phi.$$

Hence, since $\prod_V \prod_U (\mathfrak{A}_\delta, q_\delta) = \prod_{U \times V} (\mathfrak{A}_\delta, q_\delta)$, etc., we have produced a contradiction and are done.

Corollary 1. $\mathfrak{L}^{\mathrm{Top}}$, *the logic on the topological models, is maximal with respect to the Łoś ultraproducts theorem.*

Proof. A direct application of the theorem to topological models.

Corollary 2. *If* $\mathfrak{L}^* < \mathfrak{L}^{\mathfrak{F}}$ *then* $\mathfrak{L}^*$ *does not have an isomorphic ultrapowers theorem.*

Proof. Suppose $\mathfrak{L}^* < \mathfrak{L}^{\mathfrak{F}}$ and $\mathfrak{L}^*$ has an isomorphic ultrapowers theorem. Since $\mathfrak{L}^* < \mathfrak{L}^{\mathfrak{F}}$ there is a ϕ in $\mathfrak{L}^{\mathfrak{F}}$ which is not $EC_{\mathfrak{L}^*}(L)$. As in the proof of the theorem we have

$$\prod_U (\mathfrak{A}_\delta, q_\delta) \models^{\mathfrak{L}^{\mathfrak{F}}} \phi, \quad \prod_U (\mathfrak{B}_\delta, r_\delta) \not\models^{\mathfrak{L}^{\mathfrak{F}}} \phi \quad \text{and}$$

$$\prod_U (\mathfrak{A}_\delta, q_\delta) \equiv_{\mathfrak{L}^*} \prod_U (\mathfrak{B}_\delta, r_\delta)$$

for some $(\mathfrak{A}_\delta, q_\delta)$, $(\mathfrak{B}_\delta, r_\delta)$, $\delta < \lambda$. Since $\mathfrak{L}^*$ has an isomorphic ultrapowers theorem, we have a V such that $\prod_V \prod_U (\mathfrak{A}_\delta, q_\delta) \cong \prod_V \prod_U (\mathfrak{B}_\delta, r_\delta)$ which leads to a contradiction as in the theorem.

REMARK. $\mathfrak{L}(I)$, $\mathfrak{L}(I^n_{n\in\omega})$, the interior operator logics (see **[S-3]**), do not have an isomorphic ultrapowers theorem even though they satisfy interpolation since $\mathfrak{L}(I)$, $\mathfrak{L}(I^n_{n\in\omega}) < \mathfrak{L}^{\mathrm{Top}}$.

Problem. Are there any other examples of "natural" logics which have a Łoś ultraproducts theorem?

2. **Separation properties.** In this section we prove two weak separation properties for an arbitrary $\mathfrak{L}^{\mathfrak{F}}$.

DEFINITION 1. $\mathfrak{L}^*$ is said to have the *Souslin-Kleene property* iff for each $\Omega \subseteq \mathrm{Mod}$ if Ω and Ω^c are $PC_{\mathfrak{L}^*}(L)$-classes then they are $EC_{\mathfrak{L}^*}(L)$ classes.

DEFINITION 2. $\mathfrak{L}^*$ is said to have the *weak-Beth property* iff for each $\Omega \subseteq \mathrm{Mod}_{L\cup\{R\}}$, if Ω is an $EC_{\mathfrak{L}^*}(L \cup \{R\})$ class such that for each $(\mathfrak{A}, q)$ there is a unique R so that $(\mathfrak{A}, R, q) \in \Omega$, then $\{(\mathfrak{A}, a_1, \ldots, a_n, q) | (\mathfrak{A}, R, q) \in \Omega$ and $\langle a_1, \ldots, a_n\rangle \in R\}$ is an $EC_{\mathfrak{L}^*}(L \cup \{c_1, \ldots, c_n\})$ class.

THEOREM. $\mathfrak{L}^{\mathfrak{F}}$ *has the Souslin-Kleene property.*

PROOF. Suppose Ω, Ω^c are both $PC_{\mathfrak{L}^{\mathfrak{F}}}(L)$ classes, i.e. there is a $\theta_0 \in L_0^{\mathfrak{F}}$, $\theta_1 \in L_1^{\mathfrak{F}}$ such that

$$\Omega = \{(\mathfrak{A} \restriction L, \mathfrak{F}(q)) | (\mathfrak{A}, \mathfrak{F}(q)) \models^{\mathfrak{L}^{\mathfrak{F}}} \theta_0\},$$

$$\Omega^c = \{(\mathfrak{B} \restriction L, \mathfrak{F}(r)) | (\mathfrak{B}, \mathfrak{F}(r)) \models^{\mathfrak{L}^{\mathfrak{F}}} \theta_1\}.$$

To obtain a contradiction suppose that Ω is not $EC_{\mathfrak{L}^{\mathfrak{F}}}(L)$.

For each collection $0 \leqslant i \leqslant n$, $\phi_i \in L^2$ (the 2-sorted logic), we claim that there is an L_0-model $(\mathfrak{A}, q)$ and an L_1-model $(\mathfrak{B}, r)$ in Mod so that

$$(*) \qquad (\mathfrak{A}, q) \models^{\mathfrak{L}_2} \theta_0, \qquad (\mathfrak{B}, r) \models^{\mathfrak{L}_2} \theta_1,$$

and for each ϕ_i, $0 \leqslant i \leqslant n$,

$$(\mathfrak{A}, q) \models^{\mathfrak{L}_2} \phi_i \text{ iff } (\mathfrak{B}, r) \models^{\mathfrak{L}_2} \phi_i.$$

Again as in the proof of the main theorem, we suppose not and derive a contradiction.

DEFINITION 3. If $\mathfrak{D} \subseteq \mathrm{Mod}(\mathfrak{F})$, $\mathfrak{D}^* = \{(\mathfrak{A}, q) | (\mathfrak{A}, \mathfrak{F}(q)) \in \mathfrak{D}\} \subseteq \mathrm{Mod}$.

We claim $(\Omega^c)^* = (\Omega^*)^c$.

$$(\mathfrak{A}, q) \in (\Omega^c)^* \text{ iff } (\mathfrak{A}, \mathfrak{F}(q)) \in \Omega^c \text{ iff } (\mathfrak{A}, \mathfrak{F}(q)) \notin \Omega$$
$$\text{iff } (\mathfrak{A}, q) \notin \Omega^* \text{ iff } (\mathfrak{A}, q) \in (\Omega^*)^c.$$

By our assumption, we have that for each $\eta\colon n + 1 \to \{0, 1\}$ either

$$\bigcap_{0\leqslant i\leqslant n} \mathrm{Mod}(\phi_i)^{\eta(i)} \cap \Omega^* = \varnothing \quad \text{or} \quad \bigcap_{0\leqslant i\leqslant n} \mathrm{Mod}(\phi_i)^{\eta(i)} \cap (\Omega^*)^c = \varnothing.$$

Since $\bigcup_\eta \bigcap_{0\leqslant i\leqslant n} \mathrm{Mod}(\phi_i)^{\eta(i)} = \mathrm{Mod}$, we know that $\Omega^* = \mathrm{Mod}_L^{\mathfrak{L}_2}(\psi)$. We will now show that ψ is an invariant sentence of L.

$$\begin{aligned}(\mathfrak{A},q) \models \psi \text{ iff } (\mathfrak{A},q) \in \Omega^* &\text{ iff } (\mathfrak{A},\mathfrak{F}(q)) \in \Omega \\ &\text{iff } \exists \text{ an expansion } \mathfrak{A}^* \text{ of } \mathfrak{A} \text{ to } L_0 \text{ s.t. } (\mathfrak{A}^*,\mathfrak{F}(q)) \models \theta_0 \\ &\text{iff } \exists \text{ an expansion } \mathfrak{A}^{**} \text{ of } \mathfrak{A} \text{ to } L_1 \text{ s.t. } (\mathfrak{A}^{**},\mathfrak{F}(q)) \not\models \theta_1 \\ &\text{iff } (\mathfrak{A},\mathfrak{F}(q)) \notin \Omega^c \text{ iff } (\mathfrak{A},\mathfrak{F}(q)) \notin (\Omega^c)^* \\ &\text{iff } (\mathfrak{A},\mathfrak{F}(q)) \notin (\Omega^*)^c \text{ iff } (\mathfrak{A},\mathfrak{F}(q)) \in \Omega^* \\ &\text{iff } (\mathfrak{A},\mathfrak{F}(q)) \models \psi.\end{aligned}$$

But this is a contradiction since we assumed that Ω was not $EC_{\mathfrak{L}^{\mathfrak{F}}}(L)$.

Hence by (*) and the methods of our main theorem, we obtain $(\mathfrak{A}_\delta, q_\delta)$, $(\mathfrak{B}_\delta, r_\delta)$ for $\delta < \lambda$ such that

$$\prod_U (\mathfrak{A}_\delta, q_\delta) \models \theta_0, \quad \prod_U (\mathfrak{B}_\delta, r_\delta) \models \theta_1 \quad \text{and}$$

$$\prod_U (\mathfrak{A}_\delta \upharpoonright L, q_\delta) \equiv_{\mathfrak{L}_2} \prod_U (\mathfrak{B}_\delta \upharpoonright L, r_\delta).$$

By Shelah's isomorphic ultrapowers theorem we have that there is an ultrafilter V so that

$$\begin{aligned}\prod_{U\times V} (\mathfrak{A}_\delta \upharpoonright L, q_\delta) = \prod_V \prod_U (\mathfrak{A}_\delta \upharpoonright L, q_\delta) &\cong \prod_V \prod_U (\mathfrak{B}_\delta \upharpoonright L, r_\delta) \\ &= \prod_{U\times V} (\mathfrak{B}_\delta \upharpoonright L, r_\delta),\end{aligned}$$

$$\prod_{U\times V} (\mathfrak{A}_\delta, q_\delta) \models \theta_0 \quad \text{and} \quad \prod_{U\times V} (\mathfrak{B}_\delta, r_\delta) \models \theta_1 .$$

This contradicts the fact that $\Omega \cap \Omega^c = \varnothing$. So we are done.

REMARK. We say that Ω is Σ_1^1 in $\mathfrak{L}^*$ if it is the class of relativized reducts of some $\mathfrak{L}^*$-definable class; i.e., if there is a 0-morphism $L \to^\alpha K$ which is the identity except for the possibility that $\alpha(\forall) \neq \forall$, and an $\mathfrak{L}^*$-definable class Ω' of K-structures such that every $(\mathfrak{A}, q) \in \Omega'$ is α-invertible and $\Omega = \{(\mathfrak{A}, q)^{-\alpha}: (\mathfrak{A},q) \in \Omega'\}$. If we assume that $(\mathfrak{A},q)^{-\alpha} = (\mathfrak{A}^{-\alpha}, q^{-\alpha})$ where $q^{-\alpha} = \{B \cap |\mathfrak{A}^{-\alpha}| \mid B \in q\}$ and that $\mathfrak{F}(q^{-\alpha}) = (\mathfrak{F}(q))^{-\alpha}$, then the following stronger result is provable.

THEOREM. *If Ω and Ω^c are Σ_1^1 in $\mathfrak{L}^{\mathfrak{F}}$, then Ω is $EC_{\mathfrak{L}^{\mathfrak{F}}}(L)$, i.e. definable.*

The examples given in §1 all satisfy the stronger requirements.

COROLLARY. *$\mathfrak{L}^{\mathfrak{F}}$ has the weak-Beth property.*

PROOF. This is essentially the proof given in **[J]**. Assume Ω is $EC_{\mathfrak{L}^{\mathfrak{F}}}(L \cup \{R\})$ such that for each $(\mathfrak{A},q)$ there is a unique R so that $(\mathfrak{A},R,q) \in \Omega$. Let $\Omega = \mathrm{Mod}_{\mathfrak{L}^{\mathfrak{F}}}(\theta(R))$; then $\psi = \theta(R) \wedge R(c_1,\ldots,c_n)$ is an invariant sentence of $L \cup \{R, c_1, \ldots, c_n\}$. Let $\Omega^{\#} = \mathrm{Mod}(\psi) \upharpoonright L \cup \{c_1, \ldots, c_n\}$. Then $\Omega^{\#}$, $(\Omega^{\#})^c$ are both $PC_{\mathfrak{L}^{\mathfrak{F}}}(L \cup \{c_1, \ldots, c_n\})$-classes. Hence $\Omega^{\#} = \{(\mathfrak{A}, a_1, \ldots, a_n, q) | (\mathfrak{A}, R, a_1, \ldots, a_n, q) \in \Omega$ and $\langle a_1, \ldots, a_n \rangle \in R\}$ is $EC_{\mathfrak{L}^{\mathfrak{F}}}(L \cup \{c_1, \ldots, c_n\})$.

Problem. Does an arbitrary $\mathfrak{L}^{\mathfrak{F}}$ have any stronger separation properties, e.g., interpolation or isomorphic ultrapowers? $\mathfrak{L}^{\mathrm{Top}}$ has an isomorphic ultrapowers theorem.

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