MAXIMAL LOGICS

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Abstract. In this paper we present a general method for producing logics on various classes of models which are maximal with respect to a Łoś ultraproducts theorem. As a corollary we show that $\mathcal{L}_{\text{Top}}$ is maximal. We also show that these maximal logics satisfy the Souslin-Kleene property.

0. Introduction. In this paper we will prove that there is a strongest logic for certain classes of models with a Łoś ultraproduct theorem.

The motivation comes from two sources. The first is the area of abstract logic and model theory. P. Lindström first proved that $\mathcal{L}_{\omega\omega}$ is the strongest logic which satisfies the compactness and Löwenheim-Skolem theorem. K. J. Barwise [B-1] expanded, simplified, and strengthened these results by formulating abstract model theory in a category-theoretic framework.

The second area is topological model theory. In [S-1] we presented a topological logic using generalized quantifiers. This logic is formed by adding a quantifier symbol $Qx$ to $\mathcal{L}_{\omega\omega}$, denoted by $\mathcal{L}(Q)$, where the interpretation of $Qx\phi(x)$ is that the set defined by $\phi(x)$ is "open". Another logic, denoted by $\mathcal{L}(Q^n)_{n\in\omega}$, is formed by adding $Q^n x_1, \ldots, x_n$ for each $n$ so that the interpretation of $Q^n x_1, \ldots, x_n \phi(x_1, \ldots, x_n)$ is that the set defined by $\phi(x_1, \ldots, x_n)$ is "open in the $n$th product topology". However, they are not the strongest logics even though they both satisfy the compactness and Löwenheim-Skolem theorems.

More recently, S. Garavaglia and T. McKee (see [G-1] or [McK]) have found an extension, $\mathcal{L}_{\text{Top}}$, of $\mathcal{L}(Q)$ and $\mathcal{L}(Q^n)_{n\in\omega}$ which has many desirable properties, e.g. compactness, Löwenheim-Skolem, interpolation, and an isomorphic ultrapowers theorem.

Hence, one is naturally led to the question of when there is a strongest logic with first order properties. In this paper we give a construction of the strongest logic with a Łoś ultraproduct theorem. We then show that these maximal logics have the Souslin-Kleene property.

1. Abstract logics. We will assume that the reader is familiar with the basic notions of first order model theory (e.g. many-sorted logics, ultrafilters, and ultraproducts), topology and category theory.
Take a first order model $\mathfrak{A}, q \subseteq \mathfrak{F}(A)$, and form $(\mathfrak{A}, q)$. $(\mathfrak{A}, q)$ is called a weak model. (If $q$ is a topology then $(\mathfrak{A}, q)$ is called topological.) If we take a class of weak models, Mod, we can give a definition of a logic on Mod.

Our notion of a logic is similar to that of Barwise [B-1] and we will assume some familiarity with it. We take the set of objects to be $\mathcal{L}$, the class of languages. The morphisms will be the $k$-morphisms in [B-1].

We define a logic, $\mathcal{L}^*$, to consist of a syntax and a semantics. The syntax is a functor $*$ on $\mathcal{L}$. The elements $\mathcal{L}^*$ for $L \in \mathcal{L}$ are called $\mathcal{L}^*$ sentences. The functor $*$ satisfies the following axiom:

**Occurrence Axiom.** For every $\mathcal{L}^*$ sentence $\phi$ there is a smallest (under $\subseteq$) language $L_\phi$ in $\mathcal{L}$ such that $\phi \in L_\phi^*$. The semantics of $\mathcal{L}^*$ is a relation $\models_{\mathcal{L}^*}$ such that if $(\mathfrak{A}, q) \models_{\mathcal{L}^*} \phi$, then $\mathfrak{A}$ is an $L$-structure for some $L$ in $\mathcal{L}$ and $\phi \in L^*$. It satisfies the following axiom:

**Isomorphism Axiom.** If $(\mathfrak{A}, q) \models_{\mathcal{L}^*} \phi$ and $(\mathfrak{A}, q) \cong (\mathfrak{B}, r)$ (i.e. $(\mathfrak{A}, q)$, $(\mathfrak{B}, r)$ are isomorphic as 2-sorted structures), then $(\mathfrak{B}, r) \models_{\mathcal{L}^*} \phi$.

A logic on Mod which has important applications is the 2-sorted logic, $\mathcal{L}^\text{Mod}_2$. (We will write $\mathcal{L}^\text{2} \subset \mathcal{L}^\text{Mod}$. (We will write $\mathcal{L}^\text{2} \subset \mathcal{L}^\text{Mod}$ when the meaning is understood.) For $L \in \mathcal{L}$, $L^2$ is the set of 2-sorted sentences (i.e. built up from the constants, predicates, functions, individual variables, set variables, equality and $\in$ using $\lor, \land, \exists \forall, \exists X$). Note that $\in$ has its standard meaning and is a logical symbol. If $(\mathfrak{A}, q) \in \text{Mod}$ then

$$(\mathfrak{A}, q) \models_{\mathcal{L}^\text{2}} \phi$$

will be the usual satisfaction relation.

We say that $\phi$ in $\mathcal{L}^*$ is $EC_{\mathcal{L}^*}(L)$ if and only if there is a $\psi \in L^*$ such that

$$\text{Mod}^L_{\mathcal{L}^*}(\phi) = \text{Mod}^L_{\mathcal{L}^*}(\psi)$$

where

$$\text{Mod}^L_{\mathcal{L}^*}(\phi) = \{((\mathfrak{A}, q)|((\mathfrak{A}, q) \equiv_{\mathcal{L}^*} \phi, (\mathfrak{A}, q) \text{ an } L\text{-structure})\}.$$

Suppose we have two logics, $\mathcal{L}^*, \mathcal{L}^\#$, on a class of models, Mod. Then we can define an ordering between them which is a measure of their strength of expressibility. We say that $\mathcal{L}^\#$ is as strong as $\mathcal{L}^*$, $\mathcal{L}^\# \geq \mathcal{L}^*$, if for every $L^*$-sentence $\phi$ there is an $\mathcal{L}^\#$-sentence $\psi$ such that

(i) Every symbol occurring in $\psi$ occurs in $\phi$, i.e. $L_\psi \subseteq L_\phi$.

(ii) $\text{Mod}^{\mathcal{L}^\#}(\phi) = \text{Mod}^{\mathcal{L}^\#}(\psi)$.

Again taking an arbitrary class of models, Mod, let $T = \{q|(\mathfrak{A}, q) \in \text{Mod for some } \mathfrak{A}\}$ and suppose we are given an $\mathfrak{F}: T \to T$, a map, so that $\mathfrak{F}(q) = \mathfrak{F}(\mathfrak{F}(q))$ and for each $q$, if $(\mathfrak{A}, q) \in \text{Mod}$ then $(\mathfrak{A}, \mathfrak{F}(q)) \in \text{Mod}$ . We can define two logics based on this $\mathfrak{F}$. Let $\phi$ be an $\mathcal{L}_2$-sentence; then $\phi$ is called $\mathfrak{F}$-invariant if and only if for all $(\mathfrak{A}, q) \in \text{Mod},$

$$(\mathfrak{A}, q) \models_{\mathcal{L}_2} \phi \text{ if and only if } (\mathfrak{A}, \mathfrak{F}(q)) \models_{\mathcal{L}_2} \phi.$$
We define $\mathcal{L}_2^\mathcal{F}$ to be the sublogic of $\mathcal{L}_2$ (on Mod) which consists of the $\mathcal{F}$-invariant sentences. Taking $\text{Mod}(\mathcal{F}) = \{ (\mathcal{A}, \mathcal{F}(q)) | (\mathcal{A}, q) \in \text{Mod} \}$, we can define a logic, $\mathcal{L}_2^\mathcal{F}$, on $\text{Mod}(\mathcal{F})$ as follows:

(a) the $\mathcal{L}_2^\mathcal{F}$-sentences are just the $\mathcal{F}$-invariant ones,
(b) $(\mathcal{A}, \mathcal{F}(q)) \models^\mathcal{F} \phi$ if and only if $(\mathcal{A}, q) \models^\mathcal{L}_2 \phi$.

If we are given a logic $\mathcal{L}^*$ on $\text{Mod}(\mathcal{F})$ where $\mathcal{F}$ is as above then we can define an $\mathcal{L}^*_2$ on $\text{Mod}$. $\mathcal{L}^*_2$ will have the same sentences as $\mathcal{L}^*$ but the satisfaction relation will be defined as follows: if $(\mathcal{A}, q) \in \text{Mod}$ then

$$(\mathcal{A}, q) \models^\mathcal{L}^*_2 \phi \text{ if and only if } (\mathcal{A}, \mathcal{F}(q)) \models^\mathcal{L}^* \phi.$$ 

This is the analogue of $\models^\mathcal{L}^*_2$ for $\mathcal{L}^*$. We can then prove the following:

**Lemma 1.** If $\mathcal{L}^*_2 \leq \mathcal{L}_2$ then $\mathcal{L}^* \leq \mathcal{L}^\mathcal{F}$.

**Proof.** Assume $\mathcal{L}^*_2 \leq \mathcal{L}_2$ and suppose that $\phi \in \mathcal{L}^*$. Since $\phi$ is $\mathcal{E}_\mathcal{L}_2(L)$ we have $\psi \in \mathcal{L}_2$ such that $\text{Mod}^\mathcal{L}^\mathcal{F}(\phi) = \text{Mod}^\mathcal{L}_2(\psi)$ but then $\psi$ is $\mathcal{F}$-invariant so $\text{Mod}^\mathcal{L}^\mathcal{F}(\phi) = \text{Mod}^\mathcal{L}^\mathcal{F}(\psi)$.

Suppose we are given a class of models, Mod, such that if $(\mathcal{A}, q) \in \text{Mod}$, $\gamma < \lambda$ and $U$ is an ultrafilter on $\lambda$, then we have that $\prod_U (\mathcal{A}_\gamma, q_\gamma) = (\prod_U \mathcal{A}_\gamma, \prod_U q_\gamma)$ is in Mod (i.e. Mod is closed under ultraproducts, where $\prod_U \mathcal{A}_\gamma$ is the usual ultraproduct on the $\mathcal{A}_\gamma$, $\gamma < \lambda$, and $\prod_U q_\gamma = \{ \prod_U \mathcal{C}_\gamma | \mathcal{C}_\gamma \in q_\gamma \}$, where $\prod_U \mathcal{C}_\gamma = \{ (f)_\gamma | (\gamma | f(\gamma) \in \mathcal{C}_\gamma \} \in U \}$). Furthermore if $\mathcal{F}(\prod_U q_\gamma) = \mathcal{F}(\prod_U \mathcal{F}(q_\gamma))$, then we can give a notion of ultraproduct for $\mathcal{L}^*$ on $\text{Mod}(\mathcal{F})$ as follows:

$$\prod_U (\mathcal{A}_\gamma, q_\gamma) = \left( \prod_U \mathcal{A}_\gamma, \mathcal{F} \left( \prod_U q_\gamma \right) \right) \in \text{Mod}(\mathcal{F}).$$

This naturally leads to the question of whether there is an analogue to the Łoś ultraproduct theorem for $\mathcal{L}^*$ (i.e. $\prod_U (\mathcal{A}_\gamma, q_\gamma) \models^\mathcal{L}^* \phi$ if and only if $\{ \gamma | (\mathcal{A}_\gamma, \mathcal{F}(q_\gamma)) \models^\mathcal{L}^* \phi \} \in U$ for all $\phi$ in $\mathcal{L}^*$). The following lemma clarifies the situation.

**Lemma 2.** $\mathcal{L}^*_2$ has a Łoś theorem (on Mod) if and only if $\mathcal{L}^*$ has a Łoś theorem (on $\text{Mod}(\mathcal{F})$).

**Proof.** (if) Assume $\mathcal{L}^*$ is such that for all $\phi$ in $\mathcal{L}^*$, $\prod_U (\mathcal{A}_\gamma, q_\gamma) \models^\mathcal{L}^* \phi$ if and only if $\{ \gamma | (\mathcal{A}_\gamma, \mathcal{F}(q_\gamma)) \models^\mathcal{L}^* \phi \} \in U$. By the definition of $\mathcal{L}^*_2$,

$$\prod_U (\mathcal{A}_\gamma, q_\gamma) = \left( \prod_U \mathcal{A}_\gamma, \prod_U q_\gamma \right) \models^\mathcal{L}^\mathcal{F} \phi \text{ iff } \left( \prod_U \mathcal{A}_\gamma, \mathcal{F} \left( \prod_U q_\gamma \right) \right) \models^\mathcal{L}^* \phi \text{ iff } \{ \gamma | (\mathcal{A}_\gamma, \mathcal{F}(q_\gamma)) \models^\mathcal{L}^* \phi \} \in U$$

(only if) The proof is similar to the if direction.
EXAMPLES. (a) The most interesting example is topology. Let Mod\(_b\) = \{((\mathcal{A}, q) q is a base for a topology on \(A\)) and Top(\(q\)) be the topology generated by \(q\). Then

\[
\text{Top}(\text{Top}(q)) = \text{Top}(q), \quad \text{Top}\left( \prod_U q_r \right) = \text{Top}\left( \prod_U \text{Top}(q_r) \right)
\]

and \(\mathcal{C}_{\text{Top}}\) has a \(\mathcal{L}\os\) theorem.

(b) Let Mod\(_\text{Fil}\) = \{((\mathcal{A}, q) q is a base for a filter on \(A\)); then Fil(\(q\)) = the filter on \(A\) generated by \(q\). Hence Fil(\(q\)) = Fil(Fil(\(q\)), Fil(\(\prod_U q_r\)) = Fil(\(\prod_U \text{Fil}(q_r)\)) and \(\mathcal{C}_{\text{Fil}}\) satisfies a \(\mathcal{L}\os\) theorem.

(c) Let Mod\(_\text{BA}\) = \{((\mathcal{A}, q) q \subseteq \mathcal{P}(\(A\))\}. Then BA(\(q\)) (CBA(\(q\))) is the (complete) Boolean algebra generated by \(q\) and BA(\(\prod_U q_r\)) = BA(\(\prod_U \text{BA}(q_r)\)) (CBA(\(\prod_U q_r\)) = CBA(\(\prod_U \text{CBA}(q_r)\))). \(\mathcal{E}_{\text{BA}}\) and \(\mathcal{E}_{\text{CBA}}\) have \(\mathcal{L}\os\) theorems.

If \(\mathcal{E}^*\) has a \(\mathcal{L}\os\) theorem, we say that \(\mathcal{E}^*\) has the \(\mathcal{L}\os\os\) property.

The last result we need is a two-sorted version of Shelah's isomorphic ultrapowers theorem.

**Theorem.** If \((\mathcal{A}, q) \equiv_{\mathcal{E}_2} (\mathcal{B}, r)\), then there is an ultrafilter \(U\) on a cardinal \(\kappa\) such that \(\prod_U (\mathcal{A}, q) \equiv \prod_U (\mathcal{B}, r)\).

**Proof.** This two-sorted version is analogous to the proof found in [C-N].

Now we can prove

**Theorem.** Let \(\mathcal{E}^*\) be a logic on Mod(\(\mathcal{A}\)). If \(\mathcal{E}^*\) has the \(\mathcal{L}\os\os\) property, then \(\mathcal{E}_2^* \equiv \mathcal{E}^*\).

(One should note that we have not placed any restrictions on the size of \(L\).

**Proof.** Assume \(\mathcal{E}^*\) has the \(\mathcal{L}\os\os\) property. Assume furthermore that \(\mathcal{E}^* \not\equiv \mathcal{E}_2^*\) and we will proceed to produce a contradiction.

We know that \(\mathcal{E}_2^* \equiv \mathcal{E}_2\) by Lemma 1 and the assumption. Let \(\phi \in L^2\) be a sentence such that \(\phi\) is not EC\(_{\mathcal{E}_2}\)(\(L\)). Since \(L\) is a set we have that \(|L| \leq \lambda\), \(\lambda\) some infinite cardinal.

Take \(U\) to be a \(\lambda\)-regular ultrafilter on \(\lambda\) which exists. By the definition of regular, we know that there is a set \(X \subseteq U\) of power \(\lambda\) such that each \(\gamma \in \lambda\) belongs to only finitely many \(X \in \mathcal{X}\).

Let \(\{\psi_\gamma\}_{\gamma \in \lambda}\) and \(\{X_\delta\}_{\delta < \lambda}\) be enumerations of \(L^2\) and \(\mathcal{X}\), respectively. Then we claim that for each \(\delta < \lambda\) there is \((\mathcal{A}_\delta, q_\delta), (\mathcal{B}_\delta, r_\delta) \in \text{Mod}\) such that for each \(\gamma \in \Sigma(\delta)\),

\[
(\mathcal{A}_\delta, q_\delta) \not\equiv_{\mathcal{E}_2} \psi_\gamma \iff (\mathcal{B}_\delta, r_\delta) \not\equiv_{\mathcal{E}_2} \psi_\gamma
\]

and

\[
(\mathcal{A}_\delta, q_\delta) \not\equiv_{\mathcal{E}^*} \phi, (\mathcal{B}_\delta, r_\delta) \not\equiv_{\mathcal{E}^*} \phi,
\]

where \(\Sigma(\delta) = \{\gamma | \delta \in X_\gamma\}\) which is finite by the selection of \(\mathcal{X}\). Assume (*) does not hold and that \(\text{Mod}(\theta)^0 = \text{Mod}(\theta)\) and \(\text{Mod}(\theta)^1 = \text{Mod}(\theta)^{\mathcal{E}_2}\). By our assumption we know that for each \(\eta: \Sigma(\delta) \to \{0, 1\}\) either
\[ \bigcap_{\gamma \in \Sigma(\delta)} \text{Mod}(\psi_\gamma)^{\text{mod}} \cap \text{Mod}(\phi) = \emptyset \quad \text{or} \]
\[ \bigcap_{\gamma \in \Sigma(\delta)} \text{Mod}(\psi_\gamma)^{\text{mod}} \cap \text{Mod}(\phi)^c = \emptyset. \]

Then since \( \bigcup_{\gamma} \bigcap_{\gamma \in \Sigma(\delta)} \text{Mod}(\psi_\gamma)^{\text{mod}} = \text{Mod} \) and \( \Sigma(\delta) \) is finite, \( \phi \) would be \( EC_{\varepsilon_2}(L) \). This is a contradiction.

Since \( \varepsilon^* \) has the \( \text{\L o\c s} \) property, Lemma 2 implies that \( \varepsilon_{2}^* \) has the \( \text{\L o\c s} \) property, so

\[ \prod_U (\mathfrak{A}_{\delta}, q_{\delta}) \varepsilon_{2}^* \phi \quad \text{and} \quad \prod_U (\mathfrak{A}_{\delta}, q_{\delta}) \varepsilon_{2}^* \phi. \]

Also

\[ \prod_U (\mathfrak{A}_{\delta}, q_{\delta}) = \varepsilon_{2} \prod_U (\mathfrak{A}_{\delta}, q_{\delta}). \]

This follows from the following observation. If \( \psi \in L^2 \) then \( \psi = \psi_\alpha \) for some \( \alpha < \lambda \). We know from (*) that we have for each \( \delta \in \mathcal{X}_\alpha \subset U, \)

\[ (\mathfrak{A}_{\delta}, q_{\delta}) \varepsilon_{2} \psi \iff (\mathfrak{A}_{\delta}, q_{\delta}) \varepsilon_{2} \psi. \]

This yields the result.

To finish the proof we use Shelah's isomorphic ultrapowers theorem to obtain an ultrafilter, \( V \), such that

\[ \prod_U \prod_U (\mathfrak{A}_{\delta}, q_{\delta}) \approx \prod_U \prod_U (\mathfrak{A}_{\delta}, q_{\delta}). \]

Hence, since \( \prod_U \prod_U (\mathfrak{A}_{\delta}, q_{\delta}) = \prod_{U \times V} (\mathfrak{A}_{\delta}, q_{\delta}) \), etc., we have produced a contradiction and are done.

**Corollary 1.** \( \varepsilon^{\text{Top}} \), the logic on the topological models, is maximal with respect to the \( \text{\L o\c s} \) ultraproducre theorem.

**Proof.** A direct application of the theorem to topological models.

**Corollary 2.** If \( \varepsilon^* \prec \varepsilon^\rho \) then \( \varepsilon^* \) does not have an isomorphic ultrapowers theorem.

**Proof.** Suppose \( \varepsilon^* \prec \varepsilon^\rho \) and \( \varepsilon^* \) has an isomorphic ultrapowers theorem. Since \( \varepsilon^* \prec \varepsilon^\rho \) there is a \( \phi \) in \( \varepsilon^\rho \) which is not \( EC_{\varepsilon^*}(L) \). As in the proof of the theorem we have

\[ \prod_U (\mathfrak{A}_{\delta}, q_{\delta}) \varepsilon^\rho \phi, \quad \prod_U (\mathfrak{A}_{\delta}, q_{\delta}) \varepsilon^\rho \phi \quad \text{and} \]
\[ \prod_U (\mathfrak{A}_{\delta}, q_{\delta}) = \varepsilon^* \prod_U (\mathfrak{A}_{\delta}, q_{\delta}). \]

for some \( (\mathfrak{A}_{\delta}, q_{\delta}), (\mathfrak{A}_{\delta}, q_{\delta}), \delta < \lambda \). Since \( \varepsilon^* \) has an isomorphic ultrapowers theorem, we have a \( V \) such that \( \prod_V \prod_U (\mathfrak{A}_{\delta}, q_{\delta}) \approx \prod_V \prod_U (\mathfrak{A}_{\delta}, q_{\delta}) \) which
leads to a contradiction as in the theorem.

**Remark.** $\mathcal{L}(I)$, $\mathcal{L}(\mathcal{L}^{n}_{\omega})$, the interior operator logics (see [S-3]), do not have an isomorphic ultrapowers theorem even though they satisfy interpolation since $\mathcal{L}(I)$, $\mathcal{L}(\mathcal{L}^{n}_{\omega}) < \mathcal{L}^{\text{Top}}$.

**Problem.** Are there any other examples of "natural" logics which have a Łoś ultraproducts theorem?

2. **Separation properties.** In this section we prove two weak separation properties for an arbitrary $\mathcal{L}^{\mathfrak{F}}$.

**Definition 1.** $\mathcal{L}^{\mathfrak{F}}$ is said to have the Souslin-Kleene property if for each $\Omega \subseteq \text{Mod}$ if $\Omega$ and $\Omega^{c}$ are $\text{PC}_{\mathcal{L}^{\mathfrak{F}}}(L)$-classes then they are $\text{EC}_{\mathcal{L}^{\mathfrak{F}}}(L)$ classes.

**Definition 2.** $\mathcal{L}^{\mathfrak{F}}$ is said to have the weak-Beth property if for each $\Omega \subseteq \text{Mod}_{L_{U}(R)}$, if $\Omega$ is an $\text{EC}_{\mathcal{L}^{\mathfrak{F}}}(L \cup \{R\})$ class such that for each $(\mathfrak{A}, q)$ there is a unique $R$ so that $(\mathfrak{A}, R, q) \in \Omega$, then $(\mathfrak{A}, a_{1}, \ldots, a_{n}, q) | (\mathfrak{A}, R, q) \in \Omega$ and $(a_{1}, \ldots, a_{n}) \in R$ is an $\text{EC}_{\mathcal{L}^{\mathfrak{F}}}(L \cup \{c_{1}, \ldots, c_{n}\})$ class.

**Theorem.** $\mathcal{L}^{\mathfrak{F}}$ has the Souslin-Kleene property.

**Proof.** Suppose $\Omega$, $\Omega^{c}$ are both $\text{PC}_{\mathcal{L}(L)}$ classes, i.e. there is a $\theta_{0} \in L^{\mathfrak{F}}_{0}$, $\theta_{1} \in L^{\mathfrak{F}}_{1}$ such that

$$\Omega = \{(\mathfrak{A} \upharpoonright L, \mathcal{T}(q)) | (\mathfrak{A}, \mathcal{T}(q)) \models^{\mathfrak{F}} \theta_{0}\},$$

$$\Omega^{c} = \{(\mathfrak{B} \upharpoonright L, \mathcal{T}(r)) | (\mathfrak{B}, \mathcal{T}(r)) \models^{\mathfrak{F}} \theta_{1}\}.$$

To obtain a contradiction suppose that $\Omega$ is not $\text{EC}_{\mathcal{L}^{\mathfrak{F}}}(L)$.

For each collection $0 < i < n$, $\phi_{i} \in L^{2}$ (the 2-sorted logic), we claim that there is an $L_{0}$-model $(\mathfrak{A}, q)$ and an $L_{1}$-model $(\mathfrak{B}, r)$ in $\text{Mod}$ so that

$$\Omega^{*} = \{(\mathfrak{A}, q) \models^{\mathfrak{F}} \theta_{0}, \quad (\mathfrak{B}, r) \models^{\mathfrak{F}} \theta_{1},$$

and for each $\phi_{i}$, $0 < i < n$,

$$(\mathfrak{A}, q) \models^{\mathfrak{F}} \phi_{i} \iff (\mathfrak{B}, r) \models^{\mathfrak{F}} \phi_{i}.$$ Again as in the proof of the main theorem, we suppose not and derive a contradiction.

**Definition 3.** If $\mathcal{D} \subseteq \text{Mod}(\mathfrak{F})$, $\mathcal{D}^{*} = \{(\mathfrak{A}, q)(\mathfrak{A}, \mathcal{T}(q)) \in \mathcal{D}\} \subseteq \text{Mod}$.

We claim $(\Omega^{c})^{*} = (\Omega^{*})^{c}$.

$$(\mathfrak{A}, q) \in (\Omega^{c})^{*} \iff (\mathfrak{A}, \mathcal{T}(q)) \in \Omega^{c} \iff (\mathfrak{A}, \mathcal{T}(q)) \notin \Omega$$

iff $(\mathfrak{A}, q) \notin \Omega^{*} \iff (\mathfrak{A}, q) \in (\Omega^{*})^{c}$.

By our assumption, we have that for each $\eta$: $n + 1 \rightarrow \{0, 1\}$ either

$$\bigcap_{0 < i < n} \text{Mod}(\phi_{i})^{\eta(i)} \cap \Omega^{*} = \emptyset \quad \text{or} \quad \bigcap_{0 < i < n} \text{Mod}(\phi_{i})^{\eta(i)} \cap (\Omega^{*})^{c} = \emptyset.$$

Since $\bigcup_{\eta} \bigcap_{0 < i < n} \text{Mod}(\phi_{i})^{\eta(i)} = \text{Mod}$, we know that $\Omega^{*} = \text{Mod}_{\mathcal{L}^{\mathfrak{F}}}(\psi)$. We will
now show that $\psi$ is an invariant sentence of $L$.

$$(\mathfrak{A}, q) \vdash \psi \text{ iff } (\mathfrak{A}, q) \in \Omega^* \text{ iff } (\mathfrak{A}, \mathfrak{F}(q)) \in \Omega$$

iff $\exists$ an expansion $\mathfrak{A}^*$ of $\mathfrak{A}$ to $L_0$ s.t. $(\mathfrak{A}^*, \mathfrak{F}(q)) \vdash \theta_0$

iff $\exists$ an expansion $\mathfrak{A}^{**}$ of $\mathfrak{A}$ to $L_1$ s.t. $(\mathfrak{A}^{**}, \mathfrak{F}(q)) \not\vdash \theta_1$

iff $(\mathfrak{A}, \mathfrak{F}(q)) \not\in \Omega^c$ iff $(\mathfrak{A}, \mathfrak{F}(q)) \not\in (\Omega^c)^*$

iff $(\mathfrak{A}, \mathfrak{F}(q)) \not\in (\Omega^*)^c$ iff $(\mathfrak{A}, \mathfrak{F}(q)) \in \Omega^*$

iff $(\mathfrak{A}, \mathfrak{F}(q)) \vdash \psi$.

But this is a contradiction since we assumed that $\Omega$ was not $EC_{\mathfrak{F}}(L)$.

Hence by (*) and the methods of our main theorem, we obtain $(\mathfrak{A}_\delta, q_\delta)$, $(\mathfrak{B}_\delta, r_\delta)$ for $\delta < \lambda$ such that

$$\prod_U (\mathfrak{A}_\delta, q_\delta) \vdash \theta_0, \quad \prod_U (\mathfrak{B}_\delta, r_\delta) \vdash \theta_1$$

and

$$\prod_{U \times V} (\mathfrak{A}_\delta \upharpoonright L, q_\delta) = \prod_{U \times V} (\mathfrak{B}_\delta \upharpoonright L, r_\delta)$$

By Shelah's isomorphic ultrapowers theorem we have that there is an ultrafilter $V$ so that

$$\prod_{U \times V} (\mathfrak{A}_\delta \upharpoonright L, q_\delta) \models \prod_{U \times V} (\mathfrak{B}_\delta \upharpoonright L, r_\delta).$$

This contradicts the fact that $\Omega \cap \Omega^c = \emptyset$. So we are done.

**Remark.** We say that $\mathfrak{U}$ is $\mathfrak{E}^*$ in $\mathfrak{F}$ if it is the class of relativized reducts of some $\mathfrak{E}^*$-definable class; i.e., if there is a 0-morphism $L \rightarrow K$ which is the identity except for the possibility that $\alpha(\mathfrak{A}) \neq \mathfrak{A}$, and an $\mathfrak{E}^*$-definable class $\mathfrak{U}'$ of $K$-structures such that every $(\mathfrak{A}, q) \in \mathfrak{O}'$ is $\alpha$-invertible and $\Omega = \langle (\mathfrak{A}, q)^{-\alpha} : (\mathfrak{A}, q) \in \mathfrak{O}' \rangle$. If we assume that $(\mathfrak{A}, q)^{-\alpha} = (\mathfrak{A}^{-\alpha}, q^{-\alpha})$ where $q^{-\alpha} = \{ B \cap |\mathfrak{A}^{-\alpha}| : B \in q \}$ and that $\mathfrak{F}(q^{-\alpha}) = (\mathfrak{F}(q))^{-\alpha}$, then the following stronger result is provable.

**Theorem.** If $\Omega$ and $\Omega^c$ are $\Sigma^1_1$ in $\mathfrak{E}^*$, then $\Omega$ is $EC_{\mathfrak{F}}(L)$, i.e. definable.

The examples given in §1 all satisfy the stronger requirements.

**Corollary.** $\mathfrak{E}^*$ has the weak-Beth property.

**Proof.** This is essentially the proof given in [J]. Assume $\Omega$ is $EC_{\mathfrak{F}}(L \cup \{ R \})$ such that for each $(\mathfrak{A}, q)$ there is a unique $R$ so that $(\mathfrak{A}, R, q) \in \Omega$. Let $\Omega = \text{Mod}_{\mathfrak{F}}(\theta(R))$; then $\psi = \theta(R) \land R(c_1, \ldots, c_n)$ is an invariant sentence
of $L \cup \{ R, c_1, \ldots, c_n \}$. Let $\Omega^\# = \text{Mod}(\psi) \upharpoonright L \cup \{ c_1, \ldots, c_n \}$. Then $\Omega^\#$, $(\Omega^\#)^c$ are both $PC_{c^L}(L \cup \{ c_1, \ldots, c_n \})$-classes. Hence $\Omega^\# = \{(a, a_1, \ldots, a_n, q) | (a, R, a_1, \ldots, a_n, q) \in \Omega \text{ and } \langle a_1, \ldots, a_n \rangle \in R \}$ is $EC_{c^L}(L \cup \{ c_1, \ldots, c_n \})$.

**Problem.** Does an arbitrary $c^\#$ have any stronger separation properties, e.g., interpolation or isomorphic ultrapowers? $c^{\text{Top}}$ has an isomorphic ultrapowers theorem.

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