THE DECIDABILITY OF THE THEORY OF BOOLEAN ALGEBRAS WITH THE QUANTIFIER "THERE EXIST INFINITELY MANY"

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Abstract. By using the decidability of the weak second order theory of linear order we get the decidability of the theory of Boolean algebras with the additional quantifier \( Q_0 \).

The quantifier \( Q_0 \) is defined as follows:

For any \( \mathcal{A} \),

\[
\mathcal{A} \models Q_0 \varphi(x) \text{ iff } \text{card}(\{a \in \mathcal{A} \mid \mathcal{A} \models \varphi(a)\}) > \aleph_0.
\]

\( \mathcal{B} \) denotes the elementary theory of Boolean algebras, formulated in the language with the following nonlogical symbols:

One ternary predicate \( Uxyz \) (expressing the fact that \( z \) is the sum of \( x \) and \( y \)),

One binary predicate \( Cxy \) (expressing the fact that \( y \) is the complement of \( x \))

and two constants \( 0 \) and \( 1 \) (denoting respectively zero and unit).

For the sake of simplicity we assume that \( 0 \neq 1 \) is an axiom of \( \mathcal{B} \).

\( \mathcal{B}(Q_0) \) denotes the theory of Boolean algebras in the language of \( \mathcal{B} \), with the additional quantifier \( Q_0 \).

\( \mathcal{L} \) denotes the elementary theory of linear order with least element, formulated in the language with the following nonlogical symbols:

One binary predicate \( x < y \) (expressing the fact that \( x \) is less than \( y \))

and one constant \( 0 \) (denoting the least element).

\( \mathcal{L}^w \) denotes the weak second order theory of \( \mathcal{L} \) (that means, we add to the language the new variables \( X, Y, Z, \ldots \), ranging over finite sets, and the symbols \( \in, \cup, \cap, \emptyset \), denoting respectively membership relation, union, intersection and empty set).

It is known that the weak second order theory of linear order is decidable (see [4] or [5]). Thus also \( \mathcal{L}^w \) is decidable.

Let \( \mathcal{A} \) and \( \mathcal{B} \) be Boolean algebras. \( \mathcal{A} \equiv_0 \mathcal{B} \) denotes that \( \mathcal{A} \) and \( \mathcal{B} \) have the same theory in the language of Boolean algebras with the additional quantifier \( Q_0 \).

Example. Let \( \mathcal{B}_\omega \) be the Boolean algebra of finite and cofinite subsets of \( \omega \).
Then it is easily seen (for instance by using Ehrenfeucht games) that 
\( \mathcal{B}_\omega = (\mathcal{P}_\omega)^2 \). We set

\[ \varphi(x) =_d Q_0 \forall y U x y \] and \[ \psi =_d \forall x y (C x y \rightarrow \neg \varphi(x) \lor \neg \psi(y)) \].

\( \varphi(x) \) expresses that there are infinitely many elements less than \( x \) and \( \psi \) expresses that for any element \( x \), \( x \) or its complement cannot have infinitely many smaller elements. Then \( \mathcal{B}_\omega \not\models \psi \) and \( (\mathcal{P}_\omega)^2 \not\models \neg \psi \); thus we see that the theory \( \text{Ba}(Q_0) \) is more expressive than \( \text{Ba} \).

The following theorem can be found in [2] or [1, §13, Theorem 5.1]:

**Theorem 1.** Let \( \mathcal{A} \) be any Boolean algebra. Then there is a Boolean algebra \( \mathcal{B}_1 \), with \( \text{card}(\{a | a \in \mathcal{A}\}) \leq \aleph_0 \), such that \( \mathcal{A} \cong \mathcal{B}_1 \).

Let \( \mathcal{M} \) be a linearly ordered set with first element. Then \( \mathcal{Z}(\mathcal{M}) \) denotes the Boolean algebra, generated by the left-closed right-open intervals. One can show (see [3] or [6]):

**Theorem 2.** Let \( \mathcal{A} \) be a Boolean algebra with \( \text{card}(\{a | a \in \mathcal{A}\}) \leq \aleph_0 \). Then there is a linearly ordered set \( \mathcal{M} \) such that \( \mathcal{A} \cong \mathcal{Z}(\mathcal{M}) \).

Let \( \mathcal{M} \) be a linearly ordered set with first element and \( x \in \mathcal{Z}(\mathcal{M}) \). Then there are elements \( a_0, \ldots, a_n, b_0, \ldots, b_n \in \mathcal{M} \) (or \( a_0, \ldots, a_n, b_0, \ldots, b_{n-1} \in \mathcal{M} \) with \( a_0 < b_0 < \cdots < a_n < b_n \)) such that 

\[ x = \{ y | a_0 \leq y < b_0 \} \cup \cdots \cup \{ y | a_n \leq y < b_n \} \]

\[ (x = \{ y | a_0 \leq y < b_0 \} \cup \cdots \cup \{ y | a_n \leq y \}). \]

Thus \( x \) can be coded by the two disjoint finite sets \( X_0 = \{a_0, \ldots, a_n\} \) and \( X_1 = \{b_0, \ldots, b_n\} \), \( X_0 = \{a_0, \ldots, a_n\}, X_1 = \{b_0, \ldots, b_{n-1}\} \).

\( X_0 \cup X_1 \) is the support of \( x \) (denoted by \( \text{supp} \ x \)). We set

\[ \text{Sup}(X, Y) = \begin{cases} \text{Id} & X \cap Y = \emptyset \land \forall y (y \in Y \rightarrow \exists x (x \in X \land x < y)) \\ \land \forall x y (x < y \land x \in X \land y \in X \\ \rightarrow \exists z (z \in Y \land x < z \land z < y)) \\ \land \forall x y (x < y \land x \in Y \land y \in Y \\ \rightarrow \exists z (z \in X \land x < z \land z < y)); \end{cases} \]

that means, \( X \) and \( Y \) are the code of some element of the corresponding Boolean algebra. It is possible to describe union, complement, zero and unit with the help of codes.

Let \( \varphi \) be any formula of the language of \( \text{Ba}(Q_0) \). Then we have the following important fact:

\[ \mathcal{Z}(\mathcal{M}) \models Q_0 x \varphi(x) \]

iff \( \text{card}(\bigcup \{\text{supp} \ x | \mathcal{Z}(\mathcal{M}) \models \varphi(x)\}) \geq \aleph_0 \).
Now we are in the position to define a function \( * \) from the set of formulas of \( \text{Ba}(\mathbb{Q}_0) \) to the set of formulas of \( \text{LO}^w \) such that for every sentence \( \varphi \) of \( \text{Ba}(\mathbb{Q}_0) \), \( \text{Ba}(\mathbb{Q}_0) \vdash \varphi \) iff \( \text{LO}^w \vdash (\varphi)^* \).

\[
\begin{align*}
(x = y)^* &= \text{df} \ X_0 = Y_0 \land X_1 = Y_1; \\
(x = 0)^* &= \text{df} \ X_0 = \emptyset \land X_1 = \emptyset; \\
(x = e)^* &= \text{df} \ X_0 = \{\emptyset\} \land X_1 = \emptyset; \\
(Uxyz)^* &= \text{df} \ \forall x (x \in Z_0 \leftrightarrow ([x \in X_0 \land \forall y (y \in Y_0 \land y < x \rightarrow \\
\exists z (z \in Z_1 \land y < z \land z < x)]) \\
\lor [x \in Y_0 \land \forall y (y \in X_0 \land y < x \rightarrow \\
\exists z (z \in Z_1 \land y < z \land z < x)])) \\
\land \forall x (x \in Z_1 \leftrightarrow ([x \in X_1 \land \forall y (y \in Y_0 \land y < x \rightarrow \\
\exists z (z \in Z_1 \land y < z \land z < x)]) \\
\lor [x \in Y_1 \land \forall y (y \in X_0 \land y < x \rightarrow \\
\exists z (z \in Z_1 \land y < z \land z < x)])); \\
(Cxy)^* &= \text{df} \ \forall x (x \in X_0 \leftrightarrow x \in Y_1 \lor (x = \emptyset \land \emptyset \not\in Y_0)); \\
(\neg \varphi)^* &= \text{df} \ \neg \varphi^*; \\
(\varphi \land \psi)^* &= \text{df} \ \varphi^* \land \psi^*; \\
(\exists x \varphi(x))^* &= \text{df} \ \exists X_0 X_1 (\text{Sup}(X_0, X_1) \land (\varphi(x))^*); \\
(Q_0 \varphi(x))^* &= \text{df} \ \forall Y (\forall y (y \in Y \rightarrow \exists X_0 X_1 (\text{Sup}(X_0, X_1) \land \\
y \in X_0 \cup X_1 \land (\varphi(x))^*)) \\
\rightarrow \exists Z (Y \neq Z \land \forall y (y \in Y \rightarrow y \in Z) \land \\
\forall y (y \in Z \rightarrow \exists X_0 X_1 (\text{Sup}(X_0, X_1) \land y \in X_0 \cup X_1 \land (\varphi(x))^*))).
\end{align*}
\]

Let \( \varphi \) be any sentence of the language of \( \text{Ba}(\mathbb{Q}_0) \). It follows immediately from the construction, that: if \( \text{Ba}(\mathbb{Q}_0) \vdash \varphi \), then \( \text{LO}^w \vdash (\varphi)^* \). Together with Theorem 1 and Theorem 2 we also get: if \( \text{LO}^w \vdash (\varphi)^* \), then \( \text{Ba}(\mathbb{Q}_0) \vdash \varphi \). Now it follows from the decidability of \( \text{LO}^w \), that also \( \text{Ba}(\mathbb{Q}_0) \) is decidable.

**References**


