LOWER BOUNDS FOR THE ZEROS OF BESSEL FUNCTIONS

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Abstract. Let \( j_{p,n} \) denote the \( n \)th positive zero of \( J_p, p > 0 \). Then

\[
j_{p,n} > (j^2_{0,n} + p^2)^{1/2}.
\]

We begin by considering the eigenvalue problem

\[
-(xy')' + x^{-1}y = \lambda^2 x^{2p-1}y, \quad \lambda, p > 0,
\]

\[
y(a) = y(1) = 0, \quad 0 < a < 1.
\]

For simplicity of notation we will set \( q = p^{-1} \). It is easily verified that the general solution of (1) is

\[
y(x) = C_1 J_q(\lambda q x^{1/q}) + C_2 Y_q(\lambda q x^{1/q})
\]

and that the eigenvalues are given by

\[
J_q(\lambda q) Y_q(\lambda qa^{1/q}) - J_q(\lambda qa^{1/q}) Y_q(\lambda q) = 0.
\]

If \( z_n(a, r) \) denotes the \( n \)th positive zero of \( J_r(z) Y_r(z a^{1/q}) - J_r(z a^{1/q}) Y_r(z) = 0 \), then the \( n \)th eigenvalue, \( \lambda^2_n(a) \), of (1), (2) is given by

\[
\lambda^2_n(a) = (z_n(a, q)/q)^2.
\]

Let \( j_{r,n} \) denote the \( n \)th positive zero of \( J_r \). On p. 38 of [4] it is shown that \( z_n(a, r) \to j_{r,n} \) as \( a \to 0^+ \) whenever \( r \) is a positive integer. The restriction on \( r \) is extrinsic so that

\[
\lim_{a \to 0^+} z_n(a, r) = j_{r,n}, \quad r > 0.
\]

Let \( R[p, y] \) denote the Rayleigh quotient

\[
R[p, y] = \int_a^1 (-xy')' + x^{-1}y \, dx / \int_a^1 x^{2p}y^2 \, dx.
\]

It is well known that the eigenvalues \( \{\lambda^2_n(p)\} \) of (1), (2) can be obtained from the Rayleigh quotient [5]. Let \( V \) denote the linear space of all functions in \( C^2((a, 1)) \) which satisfy the boundary conditions (2). Then

\[
\lambda^2_n(p) = \min_{y \in V, y \neq 0} R[p, y].
\]

Let \( y_1, y_2, \ldots, y_n \) be \( n \) functions in \( V \), \( A \) denote the subspace of \( V \) spanned by \( y_1, y_2, \ldots, y_n \) and \( A^\perp \) denote the orthogonal complement of \( A \) relative to \( V \). Then

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\[ \lambda^2_{n+1}(p) = \max_A \min_{y \neq 0} R[p, y] \]

where the maximum is taken over all sets of \( n \) functions in \( V \).

Whenever \( p > 0 \) we have that \( x^{2p-1} < x^{-1} \) for all \( x \in (0, 1) \). Then

\[ R[p, y] = \frac{\int_a^1 - (xy')'y \, dx}{\int_a^1 x^{2p-1}y^2 \, dx} + \frac{\int_a^1 x^{-1}y^2 \, dx}{\int_a^1 x^{2p-1}y^2 \, dx} > Q[p, y] + 1, \]

where

\[ Q[p, y] = \frac{\int_a^1 - (xy')'y \, dx}{\int_a^1 x^{2p-1}y^2 \, dx} \]

is the Rayleigh quotient for the eigenvalue problem

\[ \begin{align*}
(x^2y'' + yx')' + \mu^2 x^{2p}y &= 0, \\
y(a) &= y(b) = 0,
\end{align*} \]

Equation (6) is equivalent to

\[ x^2y'' + yx' + \mu^2 x^{2p}y = 0. \]

It is easily checked that the general solution of (8) and, hence, of (6) is (recall that \( q = p^{-1} \))

\[ y(x) = C_1 J_0(\mu qx^{1/q}) + C_2 Y_0(\mu qx^{1/q}) \]

and that the eigenvalues are given by

\[ J_0(\mu q)Y_0(\mu qa^{1/q}) - J_0(\mu qa^{1/q})Y_0(\mu q) = 0. \]

In particular the \( n \)th eigenvalue, \( \mu_2^n(a) \), of (6), (7) is given by

\[ \mu_2^n(a) = (z_n(a, 0)/q)^2. \]

From (3), (5), and (9) we obtain

\[ (z_n(a, q)/q)^2 > (z_n(a, 0)/q)^2 + 1. \]

If we now replace \( q \) by \( p \), let \( a \to 0^+ \) in (10), and using (4) we find that

\[ (j_{p,n}/p)^2 > (j_{0,n}/q)^2 + 1. \]

**Theorem.** \( j_{p,n} > ((j_{0,n})^2 + p^2)^{1/2} \) whenever \( p > 0 \).

**Corollary.** \( j_{p,n} > ((n - \frac{1}{4})\pi^2 + p^2)^{1/2} \) whenever \( p > 0 \).

**Proof.** It is known (see [9, p. 489]) that the positive zeros of \( J_0 \) lie in the intervals \( (mn + \frac{3}{4} \pi, mn + \frac{5}{4} \pi) \) for \( m = 0, 1, 2, \ldots \). Hence, \( j_{0,n} > (n - 1)\pi + \frac{3}{4} \pi = (n - \frac{1}{4})\pi \). The desired result follows.

In [8] it is shown that

\[ j_{p,n} = p + a_n p^{1/3} + b_n p^{-1/3} + O(p^{-1}) \quad (n = 1, 2, \ldots), \]

where \( a_n \) and \( b_n \) are independent of \( p \). Hence,

\[ j_{p,n}^2 = p^2 + c_n p^{4/3} + O(p^{2/3}) \quad (n = 1, 2, \ldots), \]

where \( c_n \) is independent of \( p \). This shows that the second term of the lower
bound for $j_{p,n}$ given in the Theorem is of the wrong order. Other asymptotic expansions for $j_{p,n}$ may be found in [1], [2], and [6].

In [3] it is shown that for $0 \leq p \leq \frac{1}{2}$

\begin{equation}
\frac{p\pi}{2} + \left(n - \frac{1}{2}\right)\pi \leq j_{p,n}.
\end{equation}

For $p = 0$ the result of the Theorem is exact, while the expression in (11) has a strict inequality. Hence, our result is stronger than (11) whenever $p$ is sufficiently small. However, when $p = \frac{1}{2}$, the result in (11) is exact. Hence, for $0 < p < \frac{1}{2}$ neither result implies the other. It should be emphasized that the Theorem is valid for all $p > 0$, while (11) is valid only for $0 < p < \frac{1}{2}$.

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**References**


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