RELATIONSHIPS BETWEEN CONTINUUM NEIGHBORHOODS IN INVERSE LIMIT SPACES AND SEPARATIONS IN INVERSE LIMIT SEQUENCES

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Abstract. The main result of this paper is the following theorem. Let \( \{X_\alpha, f_{\alpha\beta}, \alpha, \beta \in I\} \) be an inverse system of compact Hausdorff spaces and continuous onto maps with inverse limit \( X \). Let \( p \in X \) and \( A \) be closed in \( X \). There exists a continuum neighborhood of \( p \) disjoint from \( A \) if and only if there exists \( \alpha \in I \) and disjoint sets \( U \) and \( V \) open in \( X_\alpha \), neighborhoods respectively of \( p_\alpha \) and \( A_\alpha \) such that for all \( \beta > \alpha, f_{\alpha\beta}^{-1}(U) \) lies in a single component of \( X_\beta - f_{\alpha\beta}^{-1}(V) \). This is Theorem B of the text.

Introduction. Throughout this paper the conventions and elementary results on inverse limits of topological spaces are those given in Bourbaki [3] except when noted otherwise. \((I, <)\) is a directed set, \( \{X_\alpha | \alpha \in I\} \) is a collection of nonvoid compact Hausdorff spaces and, for every pair \((\alpha, \beta)\) where \( \alpha < \beta \), \( f_{\alpha\beta} : X_\beta \to X_\alpha \) is continuous. The maps \( f_{\alpha\beta} \) are not necessarily onto. \((X_\alpha, f_{\alpha\beta})\) is an inverse system of topological spaces with inverse limit \( X \) and canonical maps \( f_\alpha : X \to X_\alpha \), for all \( \alpha \). If \( A \subseteq X \), \( p \in X \) then denote \( f_\alpha(A) = A_\alpha \) and \( f_\alpha(p) = p_\alpha \).

If \( A \) is any subset of a topological space \( S \) then \( \mathcal{F}(A) \) will denote the collection of all open (in \( S \)) subsets of \( A \) and \( \mathcal{F}(A) \) will denote the collection of all closed (in \( S \)) subsets of \( A \). \( \text{Int}(A) \) will denote the interior of \( A \) and \( \text{Clo}(A) \) will denote the closure of \( A \).

\( W \) is a subcontinuum of a topological space \( S \) if and only if \( W \) is compact, Hausdorff and connected. \( W \) is a continuum neighborhood of a point \( p \) of \( S \) if and only if \( p \) is an element of the interior of \( W \). If \( A \) is any subset of \( S \) then \( T(A) \) will denote the set of those points for which there does not exist a continuum neighborhood disjoint from \( A \).

Let \( \alpha \in I \). \( \{W_\beta | \beta > \alpha\} \) denotes an inverse system of subcontinua if and only if for all \( \beta > \alpha \), \( W_\beta \) is a subcontinuum of \( X_\beta \), and \( (W_\beta \setminus f_{\alpha\beta}^{-1}(W_\beta)) \) is an inverse system.

Let \( S \) be a compact Hausdorff space. The following dictionary provides translations of some common properties of \( S \) into properties of the set function \( T \) defined on \( \{A | A \subseteq S\} \).

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$S$ is connected im kleinem at $p \in S$ if and only if for every $A \in \mathcal{T}(S)$, if $p \not\in A$ then $p \not\in T(A)$.

$S$ is semi-locally connected at $p \in S$ if and only if $T(p) = p$.

$S$ is aposyndetic at $q$ with respect to $p$ if and only if for every $p \in S$, $T(p) = S$.

Other material on the set function $T$ can be found in [1], [2], [4], [5].

A. Fundamental theorems. Theorems A1 and A2 establish necessary and sufficient conditions on the inverse limit sequence that a given point $p$ is not an element of $T(A)$ for a given closed set $A$.

**Lemma A1.** Let $p \in X$ and $A \in \mathcal{T}(X)$. If there exists $\alpha \in I$, $U \in \mathcal{T}(X_\alpha)$, $V \in \mathcal{T}(X_\alpha)$ and an inverse system of subcontinua $\{ W_\beta \mid \beta > \alpha \}$ such that $p_\alpha \in U$, $A_\alpha \subseteq V$ and such that for all $\beta > \alpha$, $f_\alpha^{-1}(U) \subseteq W_\beta \subseteq X_\beta - f_\alpha^{-1}(V)$, then $p \not\in T(A)$.

**Proof.** Let $W$ be the canonical image of $\text{inv lim } W_\beta$ in $X$. Since each $W_\beta$ is a continuum, $W$ is a continuum. Since, for each $\beta > \alpha$, $f_\alpha^{-1}(U) \subseteq \text{Clos}(f_\alpha^{-1}(U)) \subseteq \text{Clos}(f_\alpha^{-1}(f_\alpha^{-1}(U))) \subseteq f_\beta^{-1}(W_\beta)$ and since $W = \bigcap \{ f_\beta^{-1}(W_\beta) \mid \beta > \alpha \}$, $f_\alpha^{-1}(U) \subseteq W$. Since $p_\alpha \in f_\alpha^{-1}(U)$, $p \in \text{Int}(W)$. Since, for all $\beta > \alpha$,

$$f_\beta^{-1}(W_\beta) \subseteq X - f_\beta^{-1}(V) = X - f_\alpha^{-1}(V) = f_\alpha^{-1}(X_\alpha - V) \subseteq X - A,$$

and

$$\bigcap \{ f_\beta^{-1}(W_\beta) \mid \beta > \alpha \} \subseteq X - A,$$

so $W \cap A = \emptyset$. Therefore $p \not\in T(A)$.

**Lemma A2.** Let $p \in X$ and $A \in \mathcal{T}(X)$. If $p \in T(A)$ then there exists $\alpha \in I$, $U \in \mathcal{T}(X_\alpha)$, $V \in \mathcal{T}(X_\alpha)$ and an inverse system of subcontinua $\{ W_\beta \mid \beta > \alpha \}$ such that $p_\alpha \in U$, $A_\alpha \subseteq V$ and for all $\beta > \alpha$,

$$f_\alpha^{-1}(U) \cap f_\beta(X) \subseteq W_\beta \subseteq f_\beta(X) - f_\alpha^{-1}(V).$$

**Proof.** Since $p \not\in T(A)$, there exists a subcontinuum $W$ of $X$ such that $p \in \text{Int}(W)$ and $W \cap A = \emptyset$. There exists $\alpha(p) \in I$, $U(p) \in \mathcal{T}(X_\alpha(p))$ such that $p \in f_\alpha^{-1}(U(p)) \subseteq \text{Int}(W)$ since $\{ f_\alpha^{-1}(U) \mid \alpha \in I, U \in \mathcal{T}(X_\alpha) \}$ is a basis for $\mathcal{T}(X)$. Similarly, for each $x \in A$, there exists $\alpha(x) \in I$, $U(x) \in \mathcal{T}(X_\alpha(x))$ such that $x \in f_\alpha(x)(U(x)) \subseteq X - W$. Let $\{ f_\alpha^{-1}(U_i) \mid i = 1, \ldots, n \}$ be a finite subcover of the open cover $\{ f_\alpha(x)(U(x)) \mid x \in A \}$ of the compact set $A$. Since $I$ is a directed set, there exists $\alpha \in I$ such that $\alpha \geq \alpha(p)$ and for $i = 1, \ldots, n$, $\alpha > \alpha(i)$. Let $U = f_\alpha(p)(U(p))$ and $V = f_\alpha^{-1}(U_1) \cup \cdots \cup f_\alpha^{-1}(U_n)$. For $\beta > \alpha$ let $W_\beta = f_\beta(W)$.

Let $\beta > \alpha$ and let $z \in f_\alpha^{-1}(U) \cap f_\beta(X)$. Since $z \in f_\beta(X)$, there exists $x \in X$ such that $x_\beta = z$. Since $f_\alpha(p)(z) \in U(p)$, $x_\alpha(p) \in U(p)$. Therefore $x \in f_\alpha^{-1}(U(p)) \subseteq \text{Int}(W) \subseteq W$. Thus $x = x_\beta \in f_\beta(W) = W_\beta$. So $f_\alpha^{-1}(U) \cap f_\beta(X) \subseteq W_\beta$. Now let $z \in W_\beta = f_\beta(W)$. Let $z = x_\beta$ for some $x \in W$. Clearly $z \in f_\beta(X)$. Suppose $z \in f_\alpha^{-1}(V)$. There exists $i < n$ such that $f_\alpha(i)(z) \in U_i$ so $x_\alpha(i) \in U_i$. Therefore $x \not\in W$. This contradiction establishes that $W_\beta \subseteq f_\beta(X) - f_\alpha^{-1}(V)$.  

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Theorem A1. Let \( p \in X \) and \( A \in \mathcal{F}(X) \). The following are equivalent.

(a) \( p \not\in T(A) \).

(b) There exists \( \alpha \in I, \ U \in \mathcal{F}(X_\alpha), \ V \in \mathcal{F}(X_\alpha) \) and an inverse system of subcontinua \( \{ W_\beta \mid \beta > \alpha \} \) such that \( p_\alpha \in U, \ A_\alpha \subseteq V \) and for all \( \beta > \alpha \),

\[
f_{a\beta}^{-1}(U) \cap f_\beta(X) \subseteq W_\beta \subseteq f_\beta(X) - f_{a\beta}^{-1}(V).
\]

Proof. This follows directly from Lemmas A1 and A2 once it is noted that if \( W \) is the canonical image of \( \text{inv lim} \ W_\beta \) in \( X \) then for all \( \beta, f_\beta(W) = f_\beta(X) \cap W_\beta, f_\beta(W) \) is a continuum and \( W \) is the canonical image of \( \text{inv lim} f_\beta(W) \). The relations in Lemma A1 imply that \( f_{a\beta}^{-1}(U) \cap f_\beta(X) \subseteq f_\beta(W) \subseteq f_\beta(X) - f_{a\beta}^{-1}(V) \).

Theorem A2. Suppose that each \( f_{a\beta} \) is onto. Let \( p \in X \) and \( A \in \mathcal{F}(X) \). The following are equivalent.

(a) \( p \not\in T(A) \).

(b) There exists \( \alpha \in I, \ U \in \mathcal{F}(X_\alpha), \ V \in \mathcal{F}(X_\alpha) \) and an inverse system of subcontinua \( \{ W_\beta \mid \beta > \alpha \} \) such that \( p_\alpha \in U, \ A_\alpha \subseteq V \) and such that for all \( \beta > \alpha \),

\[
f_{a\beta}^{-1}(U) \subseteq W_\beta \subseteq X_\beta - f_{a\beta}^{-1}(V).
\]

Proof. This follows directly from Theorem A1 once it is noted that, since each \( X_\alpha \) is compact, the hypothesis that each \( f_{a\beta} \) is onto forces each \( f_\alpha \) to be onto.

Corollary A1. Suppose that each \( f_{a\beta} \) is onto. Let \( J \) be a cofinal subset of \( I \) and let \( A \in \mathcal{F}(X) \). Then

\[
\cap \{ f_{a\alpha}^{-1}(T(A_\alpha)) \mid \alpha \in J \} \subseteq T(A).
\]

Proof. Suppose \( p \in X - T(A) \). Let \( \alpha, U, V \) and \( \{ W_\beta \} \) be as in Theorem A2. Since \( J \) is cofinal, there exists \( \beta \in J \) such that \( \beta > \alpha \). But then \( p_\beta \in f_{a\beta}^{-1}(U) \subseteq W_\beta \subseteq X_\beta - f_{a\beta}^{-1}(V) \subseteq X_\beta - A_\beta \). Therefore \( p_\beta \not\in T(A_\beta) \) so \( p \not\in f_\beta^{-1}(T(A_\beta)) \). Thus \( p \not\in \cap \{ f_{\gamma}^{-1}(T(A_\gamma)) \mid \gamma \in J \} \).

Corollary A2. Suppose that each \( f_{a\beta} \) is onto. Let \( J \) be a cofinal subset of \( I \) and let \( A \in \mathcal{F}(X) \). If for all \( \gamma, \delta \in I \) where \( \gamma < \delta, f_\gamma \delta \) is monotone, then

\[
T(A) = \cap \{ f_{\gamma}^{-1}(T(A_\gamma)) \mid \gamma \in J \}.
\]

Proof. Suppose \( p \in X \) and \( p \not\in \cap \{ f_{\gamma}^{-1}(T(A_\gamma)) \mid \gamma \in J \} \). There exists \( \alpha \) such that \( p_\alpha \not\in T(A_\alpha) \) so there exist \( U, V \in \mathcal{F}(X_\alpha) \) and a continuum \( W \) such that \( p_\alpha \in U \subseteq W \subseteq X_\alpha - V \subseteq X_\alpha - A_\alpha \). Now let \( \beta > \alpha \) and let \( W_\beta = f_{a\beta}^{-1}(W) \). Since \( f_{a\beta} \) is monotone, \( W_\beta \) is a continuum. It is clear that \( \{ W_\beta \mid \beta > \alpha \} \) is an inverse system of subcontinua and also that \( f_{a\beta}^{-1}(U) \subseteq W_\beta \subseteq X_\beta - f_{a\beta}^{-1}(V) \). Hence, by Theorem A2, \( p \not\in T(A) \) so \( T(A) \subseteq \cap \{ f_{\gamma}^{-1}(T(A_\gamma)) \mid \gamma \in J \} \). Com-
binning this with the above corollary gives the equality.

B. Separations in inverse limit sequences.

DEFINITION. Let $M$ and $N$ be subsets of a topological space $S$. $M$ separates $N$ in $S$ if and only if there exist sets $P$ and $Q$ such that $S - M = P \cup Q$, $P \cap N \neq \emptyset \neq Q \cap N$ and $(P \cap \text{Clo}(Q)) \cup (Q \cap \text{Clo}(P)) = \emptyset$.

REMARKS. If $N \subseteq M$ then $M$ does not separate $N$ in $S$. If $M$ does not separate $N$ in $S$ and $L \subseteq M$ then $L$ does not separate $N$ in $S$.

DEFINITIONS. Let $A \in \mathcal{F}(X)$ and $p \in X$. $p \notin S(A)$ if and only if there exists $\alpha \in I$, $U \in \mathcal{F}(X_{\alpha})$ and $v \in \mathcal{F}(X_{\alpha})$ such that $p_{\alpha} \in U$, $A_{\alpha} \subseteq V$, $U \cap V = \emptyset$, and, for all $\beta > \alpha$, $f_{\alpha \beta}^{-1}(V)$ does not separate $f_{\alpha \beta}^{-1}(U)$ in $X_{\beta}$.

REMARK. Let $A \in \mathcal{F}(X)$. It is immediate from the definitions that $A \subseteq S(A)$.

**Theorem B1.** Let $A \in \mathcal{F}(X)$. $T(A) \subseteq S(A)$.

**Proof.** Suppose $p \notin S(A)$. Let $\alpha$, $U$, $V$ be as in the above definition of $S$. Now let $\beta > \alpha$. Since $X_{\beta}$ is compact Hausdorff and $f_{\alpha \beta}^{-1}(V)$ does not separate $f_{\alpha \beta}^{-1}(U)$ in $X_{\beta}$, $f_{\alpha \beta}^{-1}(U)$ lies entirely in a single component $W_{\beta}$ of $X_{\beta} - f_{\alpha \beta}^{-1}(V)$. It is clear that $W_{\beta}$ is a subcontinuum of $X_{\beta}$, that $f_{\alpha \beta}^{-1}(U) \subseteq W_{\beta}$ and that $W_{\beta} \subseteq X_{\beta} - f_{\alpha \beta}^{-1}(V)$. For all $\beta > \alpha$, choose such a $W_{\beta}$. Now let $\gamma > \beta > \alpha$. Note that $f_{\beta \gamma}(W_{\gamma}) \subseteq f_{\beta \gamma}(X_{\gamma} - f_{\alpha \gamma}^{-1}(V)) \subseteq X_{\beta} - f_{\alpha \beta}^{-1}(V)$. Also note that $f_{\alpha \beta}^{-1}(U) = f_{\beta \gamma}(f_{\alpha \gamma}^{-1}(U)) \subseteq f_{\beta \gamma}(W_{\gamma})$ so that $f_{\alpha \beta}^{-1}(U) \subseteq f_{\beta \gamma}(W_{\gamma}) \subseteq W_{\beta}$. Since $W_{\gamma}$ is connected and $f_{\alpha \beta}^{-1}(U) \subseteq W_{\beta}$, it is clear that $f_{\alpha \beta}^{-1}(V)$ does not separate $f_{\alpha \beta}^{-1}(U)$ in $X_{\beta}$. Hence $p \notin S(A)$ and thus $S(A) \subseteq T(A)$. By Theorem B1, $S(A) = T(A)$.

The main result can now be stated.

**Theorem B.** Suppose that each $f_{\alpha \beta}$ is onto. Let $A \in \mathcal{F}(X)$, and $p \in X$. There exists a continuum neighborhood of $p$ disjoint from $A$ if and only if there exists $\alpha \in I$ and disjoint sets $U$ and $V$ open in $X_{\alpha}$, neighborhoods respectively of $p_{\alpha}$ and $A_{\alpha}$ such that for all $\beta > \alpha$, $f_{\alpha \beta}^{-1}(U)$ lies in a single component of $X_{\beta} - f_{\alpha \beta}^{-1}(V)$.

**Proof.** Once it is noted that in a compact Hausdorff space the failure of an open set $G$ to separate an open set $D$ is equivalent to the existence of a component $K$ of $S - G$ that contains $D$, the theorem follows as a corollary to Theorem B2.
Bibliography


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