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REPRESENTATIONS OF SOLVABLE LIE ALGEBRAS. IV: AN ELEMENTARY PROOF OF THE $(U/P)_E$-STRUCTURE THEOREM

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ABSTRACT. In this paper we give a shorter and much more elementary proof of a theorem which describes the structure of certain localisations of the enveloping algebra of a completely solvable Lie algebra. Such a localisation is shown to be a twisted group algebra where the group is free abelian of finite rank and the coefficient ring is a polynomial extension of a Weyl algebra.

Introduction. Let $\mathbf{g}$ be a completely solvable Lie algebra over a field k of characteristic zero, P a prime ideal of the enveloping algebra $U = U(\mathbf{g})$, E the semicentre of U/P (see §1 below) and $(U/P)_E$ the corresponding quotient ring. (The interest of $(U/P)_E$ is that any simple U-module with annihilator P is naturally a $(U/P)_E$-module and $(U/P)_E$ is a simple algebra.) If $\mathbf{g}$ is nilpotent then E is the centre of U/P and $(U/P)_E$ is a Weyl algebra A_n, $n \geqslant 0$, where

$$A_n = K[y_1, \ldots, y_n, \partial/\partial y_1, \ldots, \partial/\partial y_n],$$

the "ring of differential operators with polynomial coefficients". In [4] it was shown that if $\mathbf{g}$ is completely solvable then $(U/P)_E$ may be regarded as a ring of differential operators in which the multiplication has been altered by a 2-cocycle. In [5] the cohomology group involved was determined and it followed readily that such a "twisted" ring of differential operators had a much more elementary presentation as a "group algebra" of a free abelian group in which the group elements induce automorphisms on the coefficient ring. This group algebra is constructed (see §1) from the data (V, δ, G) (and is denoted by $\mathcal{Q}(V, \delta, G)$) where V is a finite-dimensional vector space, δ is an alternating bilinear form on V and G is a finitely generated subgroup of the additive group of the dual space V^*. The proof that $(U/P)_E$ is isomorphic to $\mathcal{Q}(V, \delta, G)$ as given in [4] and [5] is rather complicated. In particular, the proofs in [4] depend on results on smash products from [3]. In this paper we give an elementary proof that $(U/P)_E \cong \mathcal{Q}(V, \delta, G)$, which is completely independent of [3]. In order to make the whole argument intelligible we briefly sketch those

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parts of the proof which remain essentially unchanged.

1. **Preliminaries.** If M is a **g**-module then $E(M)$ is the *set* of elements of M which generate one-dimensional **g**-submodules. U or U/P will be regarded as a **g**-module via the adjoint representation [2, 2.2.21]. Thus with notation as above,

$$E(U/P) = \{x \in U/P\colon [g + P, x] \in kx \text{ for all } g \in \mathbf{g}\}.$$

If $x \in E(U/P)$ then there exists $\lambda_x \in \mathbf{g}^*$ such that $(\operatorname{ad} g)(x) = \lambda_x(g)x$ for all $g \in \mathbf{g}$. λ_x is called the *weight* of x. Also $xU/P = U/Px$ and so, since P is a prime, $E(U/P)$ is a multiplicatively closed subset of U/P. (For all of this compare [4, 2.2].) (Note that in [2, 4.3], the semicentre of U/P is the subalgebra S generated by $E(U/P)$ and not $E(U/P)$ itself.)

Let (V, δ, G) be as in the introduction. $\mathcal{Q}(V, \delta, G)$ is constructed as follows. Let $S_\delta(V)$ denote the k-algebra with 1 generated by V subject to $vw - wv = \delta(v, w)1$, $v, w \in V$. If $\lambda \in G$, the map $v \mapsto v + \lambda(v)1$, $v \in V$, extends to an automorphism θ_λ of $S_\delta(V)$. The subgroup $\{\theta_\lambda\colon \lambda \in G\}$ of $\operatorname{Aut} S_\delta(V)$ is again denoted by G and $\mathcal{Q}(V, \delta, G)$ is the twisted group algebra $S_\delta(V) \# G$. Set $V^G = \bigcap_{\lambda \in G} \operatorname{Ker} \lambda$ and $V^\delta = \{v \mid \delta(v, V) = 0\}$. If $V^G \cap V^\delta = 0$ then $\mathcal{Q}(V, \delta, G)$ is a simple algebra of [5, Theorem 4.6].

The proof of the structure theorem falls into two parts which occupy the next two sections. Throughout k is of characteristic zero and **g** is completely solvable.

2. **Reduction to the abelian by abelian case.** Let P be a prime of $U = U(\mathbf{g})$. By passing to $U(\mathbf{g}/P \cap \mathbf{g})$ we may assume that $P \cap \mathbf{g} = 0$. Let $\mathbf{n}$ be the maximum nilpotent ideal of **g** and **a** be a complementary subspace to **n** in **g**. Let π be the canonical homomorphism $U \to U/P$ and denote $\pi(U(\mathbf{n}))$ by N and the centre of N by $Z(N)$. If $N \neq Z(N)$ then by [1, 6.8] (or [4, 4.2]) there exists $e \in E \cap Z(N)$ such that $N_e \cong A_n \otimes Z(N)_e$ for some $n > 0$, $Z(N)_e$ is a finitely generated algebra and is the centre of N_e. Thus $(U/P)_e$ is generated by $A_n \otimes Z(N)_e$ and $\pi(\mathbf{a})$. If $a \in \mathbf{a}$ then $x \mapsto [\pi(a), x]$ is a derivation on $A_n \otimes Z(N)_e$ and by [4, 2.16], there exists a k-linear map $\rho\colon \mathbf{a} \to N_e = A_n \otimes Z(N)_e$ such that $\pi(a) - \rho(a)$ commutes with A_n. Let $\mathbf{a}'$ denote the subspace of U/P spanned by $\{\pi(a) - \rho(a) \mid a \in \mathbf{a}\}$. If $a_1, a_2 \in \mathbf{a}'$ then $[a_1, a_2] \in A_n \otimes Z(N)_e$ and hence to $Z(N)_e$, the centraliser of A_n in N_e. Thus $(U/P)_e$ is generated by A_n and B, where B is the subalgebra generated by $Z(N)_e \cup \mathbf{a}'$ and A_n and B commute elementwise. So $(U/P)_e \cong A_n \otimes B$ by [2, 4.6.7].

If $a' = \pi(a) - \rho(a) \in \mathbf{a}'$ then a' induces the same derivation on $Z(N)_e$ as $\pi(a)$ does. N is a union of finite-dimensional ad **g**-submodules and hence so also is $Z(N)$ and $Z(N)_e$. Thus $Z(N)_e$ is a union of finite-dimensional ad $\mathbf{a}'$-submodules. Since $[\mathbf{a}', \mathbf{a}'] \subset Z(N)_e$, there is a finite-dimensional ad $\mathbf{a}'$-submodule W of $Z(N)_e$ such that $[\mathbf{a}', \mathbf{a}'] \subset W$ and W generates the algebra $Z(N)_e$. So B is a homomorphic image of $U(\mathbf{h})$ where **h** is the solvable Lie algebra $\mathbf{h} = W + \mathbf{a}'$. Since N is a trigonalisable ad **g**-module (i.e. N is a union of finite-dimensional ad $\mathbf{g}$-modules whose Jordan-Hölder factors are one-dimensional), so also is $Z(N)$ and $Z(N)_e$ and hence $\mathbf{h}$ is completely solvable. So $B \cong U(\mathbf{h})/P'$ for some prime ideal P' and $(U/P)_e \cong A_n \otimes B \cong A_n \otimes U(\mathbf{h})/P'$. We may assume that $P' \cap \mathbf{h} = 0$.

It can happen that the maximum nilpotent ideal of $\mathbf{h}$ is nonzero (see §4). We may repeat the above process of localising and splitting off Weyl algebras. This process must stop (by a noetherian or Krull dimension argument). So there exists $e \in E(U/P)$ such that $(U/P)_e \cong A_n \otimes U(\mathbf{h})/P'$, where $\mathbf{h}$ is a completely solvable Lie algebra whose maximum nilpotent ideal is abelian.

3. **The abelian by abelian case.** From §2 it is sufficient now to consider the case when the maximum nilpotent ideal $\mathbf{n}$ of $\mathbf{g}$ is abelian and P is a prime ideal of $U(\mathbf{g})$ such that $P \cap \mathbf{g} = 0$. Consider $\mathbf{n}$ as an $\operatorname{ad}\mathbf{g}$ (or $\mathbf{g}/\mathbf{n}$)-module. $\mathbf{n}$ is a direct sum of weight spaces for $\mathbf{g}$ and we let $\lambda_1, \ldots, \lambda_t \in \mathbf{g}^*$ be the nonzero weights of $\operatorname{ad}\mathbf{g}$ on $\mathbf{n}$. (By the weight space corresponding to λ we mean the space V^λ of [2, 1.2.13].) Since $\mathbf{n}$ is the maximum nilpotent ideal, $\bigcap_{i=1}^t \operatorname{Ker} \lambda_i = \mathbf{n}$. The weights of $\operatorname{ad}\mathbf{g}$ on $U(\mathbf{g})$ are the elements of $\mathbf{N}\lambda_1 + \cdots + \mathbf{N}\lambda_t$. (See [1, 6.6].) Let $G = \mathbf{Z}\lambda_1 + \cdots + \mathbf{Z}\lambda_t$, the additive subgroup of $\mathbf{g}^*$ generated by $\lambda_1, \ldots, \lambda_t$. Since $\operatorname{char} k = 0$, G is a finitely generated torsion-free group and hence is a free abelian group whose rank will be denoted by m. For $1 \leqslant i \leqslant t$, let $e_i \in \mathbf{n}$ be an $\operatorname{ad}\mathbf{g}$-eigenvector of weight λ_i, i.e. $[g, e_i] = \lambda_i(g)e_i$ for all $g \in \mathbf{g}$. Set $e = e_1 \cdots e_t$, which is an $\operatorname{ad}\mathbf{g}$-eigenvector of weight $\lambda_1 + \cdots + \lambda_t$. No power of e belongs to P since otherwise some $e_i \in P$ (as P is prime), which contradicts $P \cap \mathbf{g} = 0$. Since $e \in E(U/P)$ we may form $(U/P)_e$. The action of $\operatorname{ad}\mathbf{g}$ extends from U/P to $(U/P)_e$ and the weights of $\operatorname{ad}\mathbf{g}$ on $(U/P)_e$ are exactly the elements of $G = \mathbf{Z}\lambda_1 + \cdots + \mathbf{Z}\lambda_t$.

Choose $\mu_1, \ldots, \mu_m \in G$ such that $G = \mathbf{Z}\mu_1 + \cdots + \mathbf{Z}\mu_m$ (recall $m = \operatorname{rank} G$) and let $E_1, \ldots, E_m$ be monomials in $\{e_1, \ldots, e_t, e_1^{-1}, \ldots, e_t^{-1}\}$ whose weights are $\mu_1, \ldots, \mu_m$, respectively. Then $E_1^{n_1} \cdots E_m^{n_m}$ ($n_i \in \mathbf{Z}$) has weight $n_1\mu_1 + \cdots + n_m\mu_m$. Thus distinct monomials in the elements of $\mathfrak{E} = \{E_1, \ldots, E_m, E_1^{-1}, \ldots, E_m^{-1}\}$ correspond to distinct weights and so the subgroup of the units of $(U/P)_e$ generated by $\mathfrak{E}$ is isomorphic to G and will also be denoted by G. (The notation of Greek letters for weights and Roman letters for elements of $(U/P)_e$ makes it clear whether we are discussing the additive group of weights or the corresponding multiplicative group of eigenvectors.) The subalgebra of $(U/P)_e$ generated by $\mathfrak{E}$ will be denoted by kG as it is the group algebra of G.

If $f \in E(U/P)$ then there exists $g \in G$ with the same weight and so there is a $c \in C = \operatorname{Centre}(U/P)_e$ such that $f = cg$. Hence the subalgebra of $(U/P)_e$ generated by $\{e_1, \ldots, e_t, e^{-1}\}$ is generated by kG and elements of C. (This gives Lemma 5.4 of [4].) Thus $(U/P)_E = ((U/P)_e)_C$. The centre of $(U/P)_E$ is K, the quotient field of C, and we consider $(U/P)_E$ as a K-algebra. Let $\mathbf{a}$ be a supplementary subspace to $\mathbf{n}$ in $\mathbf{g}$ and set $l = \dim_k \mathbf{a}$. l is also the dimension of the subspace of $\mathbf{g}^*$ spanned by the elements of G.

LEMMA 3.1. (i) *The K-subalgebra of $(U/P)_E$ generated by kG is isomorphic to KG (the group algebra of G over K).*

(ii) *The K-subspace of $(U/P)_E$ spanned by $\mathbf{a}$ has dimension l and will be denoted by $K\mathbf{a}$.*

PROOF. Let $\mu_1, \ldots, \mu_l \in G$ be a basis for the subspace of $\mathbf{g}^*$ spanned by G, $a_1, \ldots, a_l \in \mathbf{a}$ be a dual basis to the μ's and $g_1, \ldots, g_l \in G$ be $\mathbf{g}$-eigenvectors with weights $\mu_1, \ldots, \mu_l$, respectively. Then $[a_i, g_j] = \delta_{ij} g_j$. If $\sum_{i=1}^{l} k_i a_i = 0$, where k's $\in K$, then $0 = [\sum k_i a_i, g_j] = k_j g_j$ and so $k_j = 0$ which proves (ii). From (ii), the standard basis of kG over k must be linearly independent over K, which proves (i). □

Thus $(U/P)_E$ contains a subspace $K\mathbf{a}$ and a subalgebra KG and $\operatorname{ad} K\mathbf{a}$ acts faithfully as semisimple derivations on KG. We now construct another subalgebra of $(U/P)_E$ on which $\operatorname{ad}\mathbf{g}$ (or $\operatorname{ad} K\mathbf{a}$) act as locally nilpotent derivations. Let λ be a weight of $\operatorname{ad}\mathbf{g}$ on $\mathbf{n}$, $\mathbf{n}_\lambda$ the corresponding weight space and $v_1, \ldots, v_l$ a basis for $\mathbf{n}_\lambda$ such that $[g, v_i] = \lambda(g)v_i$ modulo $kv_1 + \cdots + kv_{i-1}$, for all $g \in \mathbf{g}$. Then for $i \geqslant 1$, $[g, v_1^{-1}v_i] \in kv_1^{-1}v_1 + \cdots + kv_1^{-1}v_{i-1}$. Let B be the K-subalgebra of $(U/P)_E$ generated by $\bigcup\{v_1^{-1}v_i\colon i \geqslant 2\}$, where the union is over the set of weights of ad $\mathbf{g}$ on $\mathbf{n}$. Then the elements of ad $\mathbf{g}$ or ad $K\mathbf{a}$ act as locally nilpotent derivations on B. Since every nonzero central element of $(U/P)_E$ is a unit, by [4, Lemma 5.5], $B \cup S(W)$, the symmetric algebra on a K-vector space W, such that for $a \in K\mathbf{a}$ and $w \in W$, $[a, w] \in K$ and no element of W commutes with $K\mathbf{a}$. (Thus the K-subalgebra of $(U/P)_E$ generated by $K\mathbf{a}$ and $S(W)$ is a homomorphic image of the tensor product of a Weyl algebra and a commutative polynomial algebra.)

Summarising our work so far, we have that $(U/P)_E$ is generated by the subalgebras KG and $S(W)$ and the subspace $K\mathbf{a}$. $K(N_e)$, the K-subalgebra generated by N_E $(= \pi(U(\mathbf{n}))_e)$ is generated by KG and $S(W)$ and $K\mathbf{a}$ acts faithfully as an abelian Lie algebra of semisimple derivations on KG and (possibly nonfaithfully) as an abelian Lie algebra of locally nilpotent derivations on $S(W)$. Also for $a_1, a_2 \in K\mathbf{a}$, $[a_1, a_2] \in K(N_e)$.

PROPOSITION 3.2. *Regard $S(W) \otimes KG$ as a $K\mathbf{a}$-module where the elements of $K\mathbf{a}$ act as derivations extending the given action on KG and on $S(W)$. The only ideals of $S(W) \otimes KG$ which are $K\mathbf{a}$-submodules are* 0 *and $S(W) \otimes KG$.*

PROOF. Let $g \in G$ and $\mu \in (K\mathbf{a})^*$ be the weight of g. Then the μ-weight space for $K\mathbf{a}$ on $S(W) \otimes KG$ is $S(W) \otimes g$, since each element of $K\mathbf{a}$ acts as a locally nilpotent derivation on $S(W)$. Hence, if I is a nonzero $K\mathbf{a}$-submodule of $S(W) \otimes KG$ then $I = \sum_{g \in G} I \cap (S(W) \otimes g)$, and so for some $g \in G$, $I \cap S(W) \otimes g \neq 0$. Thus, if I is an ideal then $I \cap S(W) \neq 0$. The only nonzero ideal of $S(W)$ which is a $K\mathbf{a}$-submodule is $S(W)$ itself, since $S(W) = K[y_1, \ldots, y_n]$, say, and there exist elements of $K\mathbf{a}$ acting as $\partial/\partial y_1, \ldots, \partial/\partial y_n$. □

COROLLARY 3.3. *The subalgebra of* $(U/P)_E$ *generated by* $S(W)$ *and* KG *is isomorphic to* $S(W) \otimes KG$. $\square$

Finally we consider the cohomology involved here. For $a_1, a_2 \in K\mathbf{a}$, $[a_1, a_2] \in K(N_e) = S(W) \otimes KG$ and the map: $K\mathbf{a} \times K\mathbf{a} \to S(W) \otimes KG$ defined by $(a_1, a_2) \mapsto [a_1, a_2]$ is an element of $Z^2(K\mathbf{a}, S(W) \otimes KG)$. By [5, Theorem 3.2] we may assume that this cocycle is an element of $Z^2(K\mathbf{a}, K)$.

It remains to show that $(U/P)_E$ is an $\mathcal{Q}(V, \delta, G)$ as defined in §1. Let V be the K-vector space $W \oplus K\mathbf{a} \subset (U/P)_E$. For $v_1, v_2 \in V$, $[v_1, v_2] \in K$ and $(v_1, v_2) \mapsto [v_1, v_2]$ is an alternating bilinear form δ on V. For $v \in V$ and $g \in G$, $[g, v] = \lambda_g(v) g$ where $\lambda_g \in V^*$. $\bigcap_{g \in G} \operatorname{Ker} \lambda_g = W$ and if $g_1 \neq g_2$ then $\lambda_{g_1} \neq \lambda_{g_2}$. Thus $\{\lambda_g, g \in G\}$ is a subgroup of V^* isomorphic to G. We now have the data (V, δ, G) as in §1. $V^G = W$ and $V^\delta \subseteq K\mathbf{a}$ so $V^G \cap V^\delta = 0$ and so $\mathcal{Q}(V, \delta, G)$ is a simple algebra [5, Theorem 4.6]. Now $(U/P)_E$ is generated by V and KG, so by the universal property of $\mathcal{Q}(V, \delta, G)$ [5, Theorem 4.3] there is an algebra homomorphism from $\mathcal{Q} = \mathcal{Q}(V, \delta, G)$ onto $(U/P)_E$, which is an isomorphism since $\mathcal{Q}$ is simple.

4. **An example.** We give an example to illustrate §2 and show that after splitting off the Weyl algebra arising from the maximum nilpotent ideal of $\mathbf{g}$ to get $(U/P)_E \cong A_n \otimes U(\mathbf{h})/P'$, there may be further Weyl algebras arising from $U(\mathbf{h})/P'$.

Let $\mathbf{g}$ be the six-dimensional Lie algebra with basis x, y, z, v, w, t and multiplication table, $[x,y] = z$, $[t,x] = x$, $[t,y] = -y$, $[t,v] = w$. Thus the maximum nilpotent ideal n of g is five-dimensional and is the direct sum of a three-dimensional nilpotent and a two-dimensional abelian Lie algebra.

$$U(\mathbf{n})_z = k[z, z^{-1}, v, w] \otimes A_1 \quad \text{where } A_1 = k[x, z^{-1}y].$$

$t' = t + xz^{-1}y$ commutes with A_1. The Lie algebra $\mathbf{h}$ has basis z, z^{-1}, v, w, t' and multiplication table $[t', v] = w$. So $(U(g))_{zw} \cong k[z, z^{-1}, w, w^{-1}] \otimes A_2$. $U_E \cong A_2(K)$ but the maximum nilpotent ideal of $\mathbf{g}$ contributed just one of the A_1's.

REMARK. One of the open questions of [5] (viz. whether rank δ is an invariant of $\mathcal{Q}(V, \delta, G)$) has been given an affirmative answer in [6].

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