ON THE ACCRETIVITY OF THE INVERSE OF AN ACCRETIVE RELATION

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Abstract. If $X$ is a smooth, reflexive, real Banach space such that a relation $A$ in $X \times X$ is accretive iff $A^{-1}$ is accretive, then $X$ is isomorphic to a Hilbert space.

Let $X$ be a real Banach space with dual $X^*$. A subset $A \subset X \times X$ is said to be accretive if for all $[x_1, y_1], [x_2, y_2] \in A$, $i = 1, 2$, $(y_1 - y_2, f) \geq 0$ for some $f \in F(x_1 - x_2)$, where $F$ is the duality map: $F(x) = \{x^* \in X^*|(x, x^*) = \|x\|^2 = \|x^*\|^2\}$. By definition $A^{-1} = \{(x, y) | [y, x] \in A\}$. It is trivial that if $X$ is a real Hilbert space then $A$ is accretive (or monotone) iff $A^{-1}$ is accretive.

Theorem. Suppose $X$ is a smooth, reflexive, real Banach space such that $A$ is accretive in $X \times X$ iff $A^{-1}$ is accretive in $X \times X$. Then $X$ is isomorphic to a Hilbert space.

Proof. As $X$ is smooth and reflexive, the duality map $F$ is single-valued. Define $\langle x, y \rangle = \langle x, F(y) \rangle$, $x, y \in X$. It follows from the assumption that $\langle x, y \rangle \geq 0$ iff $\langle y, x \rangle \geq 0$, and so also that

\[ \langle x, y \rangle = 0 \iff \langle y, x \rangle = 0 \quad \forall x, y \in X. \]

For any closed subspace $M$ of $X$ define

\[ M^\perp = \{ x \in X | \langle x, y \rangle = 0 \ \forall y \in M \}. \]

By (1) it is seen that $M$ is a closed subspace of $X$. If $x \in M \cap M^\perp$ then $\|x\|^2 = \langle x, x \rangle = 0$, i.e., $x = 0$; and so

\[ M \cap M^\perp = \{0\}. \]

Let $z \in X$ be arbitrary. Let $x_1$ be an element in $M$ that minimizes the norm $\|z - x\|$, $x \in M$ ($x_1$ exists because $M$ is closed and convex and $X$ is reflexive). This implies that $\|z - x_1\| \leq \|z - x_1 + y\| \forall y \in M$. Lemma 1.1 in [1] now yields $\langle y, z - x_1 \rangle = 0 \forall y \in M$, and so $z - x_1 \in M^\perp$. It follows that

\[ X = M + M^\perp, \]

which together with (3) and the fact that $M^\perp$ is a closed subspace shows that $M$ has a topological complement. As $M$ was an arbitrary closed subspace of $X$, Theorem 1 in [2] implies that $X$ is isomorphic to a Hilbert space.
REFERENCES


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