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EQUIPARTITION OF ENERGY FOR A CLASS OF SECOND ORDER EQUATIONS

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ABSTRACT. We consider the Cauchy problem for a class of second order equations of the form $(d/dt - A_2)(d/dt - A_1)u(t) = 0$ in a Hilbert space H. A d'Alembert type solution formula is presented and we give a suitable definition of energy. Also, we derive a necessary and sufficient condition for the asymptotic equipartition of energy (Kinetic and Potential) to hold. These results generalize corresponding results for the abstract wave equation $(d^2/dt^2 + A^2)u(t) = 0$.

1. **Introduction.** If $a_1 \neq a_2$ are complex numbers, the general solution of the ordinary differential equation

$$u'' - (a_1 + a_2)u' + a_1 a_2 u = 0 \tag{$*$}$$

is given by

$$\begin{aligned} u(t) &= \exp(ta_1)c_1 + \exp(ta_2)c_2 \\ &= \tfrac{1}{2}[\exp(ta_1) + \exp(ta_2)](c_1 + c_2) \\ &\quad + \tfrac{1}{2}[\exp(ta_1) - \exp(ta_2)](c_1 - c_2), \end{aligned}$$

where c_1, c_2 are arbitrary constants. In addition, if initial data $u(0) = u_0$, $u'(0) = u_1$ are prescribed, then c_1 and c_2 can be determined from the two equations $c_1 + c_2 = u_0$, $a_1c_1 + a_2c_2 = u_1$ and, hence, the unique solution of $(*)$ in this case is given by the formula

$$\begin{aligned} u(t) &= \tfrac{1}{2}[\exp(ta_1) + \exp(ta_2)]u_0 \\ &\quad + [\exp(ta_1) - \exp(ta_2)](a_1 - a_2)^{-1}\left(u_1 - \frac{a_1 + a_2}{2}u_0\right). \end{aligned} \tag{1}$$

Notice that, in view of the identity

$$\exp(ta_1) - \exp(ta_2) = \int_0^t \exp((t-s)a_1)\exp(sa_2)(a_1 - a_2)\,ds, \tag{2}$$

we can rewrite (1) as

$$\begin{aligned} u(t) &= \tfrac{1}{2}[\exp(ta_1) + \exp(ta_2)]u_0 \\ &\quad + \int_0^t \exp((t-s)a_1)\exp(sa_2)\left(u_1 - \frac{a_1 + a_2}{2}u_0\right)ds, \end{aligned} \tag{3}$$

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this latter expression having the advantage that it is meaningful even if $a_1 = a_2$.

This d'Alembert type formula suggests a similar one for the Cauchy problem associated with the abstract equation

$$(d/dt - A_2)(d/dt - A_1)u(t) = 0 \tag{4}$$

in a Hilbert space H. We shall do this in the next section, where we also give a definition of "energy" and show it is preserved for all times. In §3, a necessary and sufficient condition in order to have equipartition of energy is given. Similar results for the wave equation ($A_1 = -A_2 = iA, A$ selfadjoint) can be found in [1], [2], [5].

2. **Representation of a solution and energy.** Let A_1, A_2 be skewadjoint operators on a Hilbert space H (with scalar product $(\cdot, \cdot)$ and norm $\|\cdot\|$) and $\{T_1(t) | \ t \in \mathbf{R}\}$, $\{T_2(t) | \ t \in \mathbf{R}\}$ the unitary groups they generate, respectively. We assume that

$$T_1(t)T_2(s) = T_2(s)T_1(t) \tag{5}$$

for all $t, s \in \mathbf{R}$. Then it is easy to see that an analogue of (2) holds, namely,

$$[T_1(t) - T_2(t)]v = \int_0^t T_1(t-s)T_2(s)(A_1 - A_2)v\,ds \quad \forall v \in \mathfrak{D}(A_1) \cap \mathfrak{D}(A_2). \tag{6}$$

From now on, we shall assume that $\mathfrak{D}_\infty = \bigcap_{k=1}^\infty \mathfrak{D}_k$ is dense in H, where $\mathfrak{D}_k = \bigcap_{j_i \in \{1,2\}} \mathfrak{D}(A_{j_1} \cdots A_{j_k})$, $k = 1, 2, \ldots$, and we make the following

DEFINITION. A function u: $\mathbf{R} \to H$ is a solution of (4) if $u \in C^1(\mathbf{R}, H) \cap C^0(\mathbf{R}, \mathfrak{D}(A_1))$ and $v_1 = du/dt - A_1 u \in C^1(\mathbf{R}, H) \cap C^0(\mathbf{R}, \mathfrak{D}(A_2))$ with $dv_1/dt - A_2 v_1(t) = 0 \ \forall t \in \mathbf{R}$.

Then the Cauchy problem for (4) with initial data $u(0) = u_0 \in \mathfrak{D}_2$, $u'(0) = u_1 \in \mathfrak{D}_1$ has its solution (it is clearly unique) given by the formula

$$u(t) = \tfrac{1}{2}[T_1(t) + T_2(t)]u_0 + \int_0^t T_1(t-s)T_2(s)\left(u_1 - \frac{A_1 + A_2}{2}u_0\right)ds. \tag{7}$$

Indeed, since $u_0 \in \mathfrak{D}_2$, $u_1 \in \mathfrak{D}_1$ and in view of (5), we have that

$$u \in C^1(\mathbf{R}, H)$$

with

$$\begin{aligned} u'(t) &= \tfrac{1}{2}[T_1(t)A_1 + T_2(t)A_2]u_0 \\ &\quad + \int_0^t T_1(t-s)T_2(s)A_1\left(u_1 - \frac{A_1 + A_2}{2}u_0\right)ds \\ &\quad + T_2(t)\left(u_1 - \frac{A_1 + A_2}{2}u_0\right). \end{aligned} \tag{8}$$

Also, $u \in C^0(\mathbf{R}, \mathfrak{D}(A_1))$, with

$$A_1 u(t) = \tfrac{1}{2}[T_1(t)A_1 + T_2(t)A_1]u_0$$
$$+ \int_0^t T_1(t-s)T_2(s)A_1\left(u_1 - \frac{A_1 + A_2}{2}u_0\right) ds,$$

and hence, $v_1(t) = u'(t) - A_1 u(t) = T_2(t)(u_1 - A_1 u_0)$. Again, our hypothesis yields $v_1 \in C^1(\mathbf{R}, H)$ with $v_1'(t) = T_2(t)A_2(u_1 - A_1 u_0)$ and

$$v_1 \in C^0(\mathbf{R}, \mathfrak{D}(A_2))$$

with $A_2 v_1(t) = T_2(t)A_2(u_1 - A_1 u_0)$ and, hence, $v_1'(t) - A_2 v_1(t) = 0$ $\forall t \in \mathbf{R}$. Finally, the initial conditions $u(0) = u_0$, $u'(0) = u_1$ can be easily verified from (7) and (8).

Now, it is also easy to check that the same formula (7) gives the solution of $(d/dt - A_1)(d/dt - A_2)u(t) = 0$, $u(0) = u_0 \in \mathfrak{D}_2$, $u'(0) = u_1 \in \mathfrak{D}_1$, so that the order in which we write the factors in (4) does not matter.[1]

Therefore, if $u(t)$ is a solution of (4), then $v_1(t) = u'(t) - A_1 u(t)$ and $v_2(t) = u'(t) - A_2 u(t)$ satisfy the first-order equations $v_1'(t) = A_2 v_1(t)$ and $v_2'(t) = A_1 v_2(t)$, respectively, and hence,

$$\|v_1(t)\| = \|v_1(0)\|, \qquad \|v_2(t)\| = \|v_2(0)\| \qquad \forall t \in \mathbf{R}. \tag{9}$$

Now, defining the operators $B = (A_1 + A_2)/2$ and $D = (A_1 - A_2)/2$, we can rewrite (9) as

$$\left.\begin{aligned} \|(u'(t) - Bu(t)) - Du(t)\|^2 &= \|(u'(0) - Bu(0)) - Du(0)\|^2 \\ \|(u'(t) - Bu(t)) + Du(t)\|^2 &= \|(u'(0) - Bu(0)) + Du(0)\|^2 \end{aligned}\right\} \quad \forall t \in \mathbf{R}. \tag{10}$$

Applying the Parallelogram Law to (10) yields

$$\begin{aligned} &2\{\|u'(t) - Bu(t)\|^2 + \|Du(t)\|^2\} \\ &\quad = 2\{\|u'(0) - Bu(0)\|^2 + \|Du(0)\|^2\} \qquad \forall t \in \mathbf{R}. \end{aligned} \tag{11}$$

Therefore, we are naturally led to define "the energy at time t" of a solution of (4) by the formula

$$E(t) = \left\|u'(t) - \frac{A_1 + A_2}{2}u(t)\right\|^2 + \left\|\frac{A_1 - A_2}{2}u(t)\right\|^2, \tag{12}$$

and we have proved

THEOREM 1. *Assume* (5). *Then the Cauchy problem for* (4) *with initial data* $u(0) = u_0 \in \mathfrak{D}_2$, $u'(0) = u_1 \in \mathfrak{D}_1$ *has a unique solution given by* (7). *Furthermore, its total energy* (12) *is preserved,*

$$E(t) = E(0) = \left\|u_1 - \frac{A_1 + A_2}{2}u_0\right\|^2 + \left\|\frac{A_1 - A_2}{2}u_0\right\|^2 \qquad \forall t \in \mathbf{R}.$$

[1] However, we are not allowed to write $d^2u/dt^2 - (A_1 + A_2)du/dt + A_1A_2u(t) = 0$, since according to our definition of solution, it need not be twice differentiable, necessarily.

REMARKS. (1) In the case of the wave equation $(d^2/dt^2 + A^2)u(t) = 0$, we have $A_1 = -A_2 = iA$ (A selfadjoint), $T_2(t) = T_1(-t)$, so that (5) is trivially satisfied and (7) reads

$$u(t) = \tfrac{1}{2}[T_1(t) + T_1(-t)]u_0 + \int_0^t T_1(t - 2s)u_1\, ds, \tag{13}$$

or,

$$u(t) = \tfrac{1}{2}[T_1(t) + T_1(-t)]u_0 + \frac{1}{2}\int_{-t}^{t} T_1(s)u_1\, ds. \tag{14}$$

R. Hersh obtained this d'Alembert type formula (among many others for higher-order equations) by a different method in [3]. Also, our definition of "energy" (12) coincides with the usual definition (see [1], [2], [5]) since $(A_1 + A_2)/2 = 0$ and $(A_1 - A_2)/2 = iA$.

(2) Still in the case of the wave equation, we observe that, if u_1 is in the range of A, (6) enables us to write (13) as

$$u(t) = \tfrac{1}{2}[T_1(t) + T_1(-t)]u_0 + \tfrac{1}{2}[T_1(t) - T_1(-t)](iA)^{-1}u_1,$$

or,

$$u(t) = \tfrac{1}{2}[T_1(t)(u_0 + (iA)^{-1}u_1) + T_1(-t)(u_0 - (iA)^{-1}u_1)], \tag{15}$$

a well-known formula derived by Hille [4]. Of course (as Hersh observes in [3]), (14) has the advantage that it is meaningful even if u_1 is not in the range of A.

(3) Clearly, we do not have to assume (5) in Theorem 1 in order to obtain an existence and uniqueness theorem. That assumption is made to give us conservation of "energy", as defined by (12).

3. **Asymptotic equipartition of energy.** We start by defining the "Kinetic" and "Potential" energies in (12) by

$$K(t) = \left\| u'(t) - \frac{A_1 + A_2}{2}u(t) \right\|^2 \quad \text{and} \quad P(t) = \left\| \frac{A_1 - A_2}{2}u(t) \right\|^2,$$

respectively. Still assuming (5) and that $u(0) = u_0 \in \mathfrak{D}_2$, $u'(0) = u_1 \in \mathfrak{D}_1$, we get from (7) that

$$\begin{aligned}(A_1 - A_2)u(t) &= \tfrac{1}{2}[T_1(t) + T_2(t)](A_1 - A_2)u_0 \\ &\quad + \int_0^t T_1(t - s)T_2(s)(A_1 - A_2)\left(u_1 - \frac{A_1 + A_2}{2}u_0\right) ds \\ &= \tfrac{1}{2}[T_1(t) + T_2(t)](A_1 - A_2)u_0 + [T_1(t) - T_2(t)]\left(u_1 - \frac{A_1 + A_2}{2}u_0\right) \\ &= T_1(t)(u_1 - A_2 u_0) - T_2(t)(u_1 - A_1 u_0),\end{aligned}$$

the second equality holding in view of (6). Therefore, the "potential energy" is given by

$$\begin{aligned} P(t) &= \left\| \frac{A_1 - A_2}{2} u(t) \right\|^2 \\ &= \tfrac{1}{4}[\|u_1 - A_2 u_0\|^2 + \|u_1 - A_1 u_0\|^2 \\ &\qquad -2 \operatorname{Re}(T_1(t)(u_1 - A_2 u_0), T_2(t)(u_1 - A_1 u_0))]. \end{aligned}$$

Now, by the same argument used in §2, we have

$$\begin{aligned} \|u_1 - A_2 u_0\|^2 + \|u_1 - A_1 u_0\|^2 &= 2\left(\left\| u_1 - \frac{A_1 + A_2}{2} u_0 \right\|^2 + \left\| \frac{A_1 - A_2}{2} u_0 \right\|^2 \right) \\ &= 2E(0) \end{aligned}$$

and hence,

$$\begin{aligned} P(t) &= \tfrac{1}{4}[2E(0) - 2 \operatorname{Re}(T_1(t)(u_1 - A_2 u_0), T_2(t)(u_1 - A_1 u_0))] \\ &= \tfrac{1}{2}E(0) - \tfrac{1}{2} \operatorname{Re}(T_1(t)(u_1 - A_2 u_0), T_2(t)(u_1 - A_1 u_0)). \end{aligned}$$

This proves

THEOREM 2. *If* (5) *holds and* $u(t)$ *is the solution of* (4) *with data* $u(0) = u_0 \in \mathfrak{D}_2$, $u'(0) = u_1 \in \mathfrak{D}_1$, *then,*

$$\lim_{|t| \to \infty} K(t) = \lim_{|t| \to \infty} P(t) = \tfrac{1}{2}E(0)$$

if and only if

$$\lim_{|t| \to \infty} \operatorname{Re}(T_1(t)(u_1 - A_2 u_0), T_2(t)(u_1 - A_1 u_0)) = 0.$$

Now, a few comments on hypothesis (5) are in order. It means that the selfadjoint operators A_1/i and A_2/i "permute", i.e., that their spectral families commute. Therefore, by a theorem of von Neumann [6, Chapter IX], both A_1/i and A_2/i are functions of some selfadjoint operator A,

$$A_1 = i\varphi_1(A), \qquad A_2 = i\varphi_2(A).$$

Let $\{E_\lambda\}$ denote the spectral family associated with A. Since $T_1(t) = \exp(tA_1) = \exp(it\varphi_1(A))$ and $T_2(t) = \exp(tA_2) = \exp(it\varphi_2(A))$, we can write

$$\begin{aligned} (T_1(t)h, T_2(t)h) &= \int_{-\infty}^{\infty} e^{it\varphi_1(\lambda)} d(E_\lambda h, T_2(t)h) \\ &= \int_{-\infty}^{\infty} e^{it\varphi_1(\lambda)} d_\lambda \left(\int_{-\infty}^{\infty} \overline{e^{it\varphi_2(\mu)}} d_\mu (E_\lambda h, E_\mu h) \right) \\ &= \int_{-\infty}^{\infty} e^{it\varphi_1(\lambda)} d_\lambda \left(\int_{-\infty}^{\lambda} e^{-it\varphi_2(\mu)} d_\mu (E_\mu h, h) \right) \\ &= \int_{-\infty}^{\infty} e^{it[\varphi_1(\lambda) - \varphi_2(\lambda)]} d\|E_\lambda h\|^2 \qquad \forall h \in H. \end{aligned}$$

On the other hand, since $\mathfrak{D}_1$ is dense in H, the requirement that the condition

$$\lim_{|t|\to\infty} \operatorname{Re}(T_1(t)(u_1 - A_2 u_0), T_2(t)(u_1 - A_1 u_0)) = 0$$

of Theorem 2 hold for all $u_0 \in \mathfrak{D}_2$, $u_1 \in \mathfrak{D}_1$ is clearly equivalent to $\lim_{|t|\to\infty}(T_1(t)h, T_2(t)h) = 0$ $\forall h \in H$. Hence, by what we saw above, one has equipartition of energy for any solution $u(t)$ of (4) with $u(0) \in \mathfrak{D}_2$, $u'(0) \in \mathfrak{D}_1$ if and only if

$$(16) \qquad \int_{-\infty}^{\infty} e^{it[\varphi_1(\lambda)-\varphi_2(\lambda)]} d\|E_\lambda h\|^2 = 0 \qquad \forall h \in H.$$

We observe that in the special case of the wave equation, we have $A_1 = -A_2 = iA$ (A selfadjoint) so that we choose $\varphi_1(\lambda) = \lambda$, $\varphi_2(\lambda) = -\lambda$, and (16) reads

$$\lim_{|t|\to\infty} \int_{-\infty}^{\infty} e^{2it\lambda} d\|E_\lambda h\|^2 = 0 \qquad \forall h \in H,$$

which is the condition presented in [2].

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