STIEFEL-WHITNEY HOMOLOGY CLASSES OF QUASI-REGULAR CELL COMPLEXES

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Abstract. A quasi-regular cell complex is defined and shown to have a reasonable barycentric subdivision. In this setting, Whitney's theorem that the k-skeleton of the barycentric subdivision of a triangulated n-manifold is dual to the (n - k)th Stiefel-Whitney cohomology class is proven, and applied to projective spaces, lens spaces and surfaces.

1. QR complexes. A (finite) cell structure on a space X is defined (see, e.g., [Brown, p. 124]) to be a (finite) family of maps \( \{ \sigma : E^n \to X \} \) called cells so that

(i) \( X = \bigcup \sigma(\text{int}(E^n)) \).
(ii) \( \sigma|\text{int}(E^n) \) is a homeomorphism onto its image.
(iii) \( \sigma(\partial E^n) \subset \bigcup \mu(E^n) \).

(We will deal with finite complexes throughout for simplicity, although everything holds in the locally finite context.) The cell structure will be called quasi-regular (QR) if the following condition holds: for each cell \( \sigma : E^n \to X \), there is a cell structure (necessarily unique) on \( \partial E^n \) so that for each cell \( \omega \) in \( \partial E^n \), \( \sigma \circ \omega \) is a cell in \( X \); note that this boundary structure must also be QR. Such a structure will be called a QR complex and \( \tilde{X} \) will denote the space \( X \) with this additional structure. The most familiar example of a QR structure is the usual cell structure on \( \mathbb{R}P^n \), denoted \( \mathbb{R}P^n \), with one cell in each dimension. Also, any regular cell complex is QR.

The product of two QR structures is defined as usual by taking for cells in the product \( X \times Y \), products \( \{ \sigma \times \tau : E^n \times E^n \to X \times Y \} \) of cells \( \sigma \) and \( \tau \) in \( X \) and \( Y \) respectively, a structure which is easily checked to be QR. Barycentric subdivision can be defined inductively in analogy with the simplicial case as follows: for each \( n \)-cell \( \sigma \), cone over the cells in the subdivision of the associated QR structure on \( \partial E^n \) and consider the set of cells so obtained: a QR complex with only zero cells is its own subdivision. It is not hard to see that the subdivision of a QR complex \( \tilde{X} \) is a QR complex, denoted \( \tilde{X}' \), and is in fact a pseudo-triangulation in the sense of [Hilton and Wiley], so that a second subdivision yields a triangulation. Incidence numbers can also be defined for a QR complex: if \( \sigma \) and \( \tau \) are two cells in \( X \), then \( [\tau, \sigma] \) is the number of cells \( \omega \) in the QR structure on \( \partial E^n \) corresponding to \( \tau \) so that \( \tau \circ \omega = \sigma \). In the barycentric subdivision, the origins of the original

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cells are the vertices of the pseudo-triangulation and \([\tau, \sigma]\) is the number of 1-cells spanned by the origins of \(\tau\) and \(\sigma\) (if \(\dim(\tau) > \dim(\sigma)\)).

**Definition.**

\[
X_0 = \bigcup \left\{ \sigma(\text{int}(E^\tau)) \mid \sum_{\tau} [\tau, \sigma] \equiv 0 \pmod{2} \right\},
\]

\[
X_1 = \bigcup \left\{ \sigma(\text{int}(E^\tau)) \mid \sum_{\tau} [\tau, \sigma] \equiv 1 \pmod{2} \right\}.
\]

Recall that \(\sum \text{rank}(H(X, X - x; \mathbb{Z}_2))\) is the local Euler number of \(X\) at \(x\).

**Theorem 1.**

\[
X_0 = \{ x \mid \text{local Euler number of } X \text{ at } x \equiv 1 \pmod{2} \},
\]

\[
X_1 = \{ x \mid \text{local Euler number of } X \text{ at } x \equiv 0 \pmod{2} \}.
\]

The decomposition \(X = X_0 \cup X_1\) thus depends only on \(X\) and not on the \(QR\) structure.

**Proof.** A straightforward proof, left to the reader, shows that if \(\tilde{X}\) is a triangulation, then \(\sum [\tau, \sigma]\) and \(1 + \sum \text{rank}(H^*(X, X - x; \mathbb{Z}_2))\) are both equal to the Euler characteristic of the link of \(x\) in \(\tilde{X}\), mod 2. The following claim will then establish the general case, since two barycentric subdivisions of a \(QR\) structure yield a triangulation.

**Claim.** The decomposition \(X = X_0 \cup X_1\) is invariant under barycentric subdivision.

It is now necessary to introduce some notation for barycentric subdivision that is a direct generalization of the usual for a simplicial complex. With each \(k\)-cell \(\mu\) in \(\tilde{X}\) we associate a sequence of cells in \(\tilde{X}\) as follows: if \(k = 0\), then \(\sigma_0\) is the unique cell whose interior contains \(\mu\); if \(k > 0\), then there is a unique \(n\)-cell \(\sigma_k\) so that \(\mu\) is the image under \(\sigma_k\) of the cone over some cell \(\omega\) in \(\partial E^n\). The following claim will then establish the general case, since two barycentric subdivisions of a \(QR\) structure yield a triangulation.

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**Claim.** The decomposition \(X = X_0 \cup X_1\) is invariant under barycentric subdivision.
Step 2. The number of \( \eta > \mu \) obtained by replacing \( \sigma_0 \) with \( \tau^a \sigma_0 \) is even.

The number of such \( \eta \) is the number of cells in the structure on \( \partial E^{\dim(a)} \) which is \( \chi(\partial E^{\dim(a)}) \mod 2 \) and thus even.

Step 3. The number of \( \eta > \mu \) obtained by replacing \( \sigma_{i-1} \sigma_i \) with \( \alpha_{i-1} \tau^b \sigma_i \) is even.

We are counting the number of ways of choosing the dotted maps below to make the diagram commute.

\[
\begin{array}{c}
\partial E^{\dim(a)} \\
\downarrow \beta \\
E^{\dim(a)} \\
\downarrow \alpha \\
E^{\dim(\tau)} \\
\downarrow \tau \\
X \\
\end{array}
\]

This is the same as choosing \( \alpha \) and \( \beta \), of which there are \( \sum \beta, \omega \) choices (the sum taken in \( \partial E^{\dim(a)} \)). By induction, \( (\partial E^{\dim(a)})_0 = \partial E^{\dim(a)} \) so this sum is even.

Step 4. The number of \( \eta \supset \mu \) is equivalent \( \mod 2 \) to \( \sum [\tau, \sigma_k] \).

To choose \( \eta \) requires 0 or 1 choice as in Step 1 and some number of choices as in Steps 2 and 3, making at least one choice. This completes the proof of the claim and the theorem. \( \square \)

Definition. Let \( C_p(X) \) be the \( p \)-chain which is the sum of the \( p \)-cells of the barycentric subdivision \( \tilde{X} \). \( \tilde{X} \) is said to be an Euler \( \mod 2 \) space if \( C_p(\tilde{X}) \) is a \( p \)-cycle for each \( p \), in which case we define \( \omega_p(\tilde{X}) = [C_p(\tilde{X})] \in H_p(X; \mathbb{Z}_2) \).

Theorem 2. \( \partial (C_{k+1}(\tilde{X})) = \sum \{\tau|\tau \text{ is a } k \text{-cell in } \tilde{X} \text{ whose interior lies in } X_1\} \).

Corollary 1. \( X \) is Euler \( \mod 2 \) if and only if \( X = X_0 \). The property of being Euler \( \mod 2 \) is therefore a topological property.

Corollary 2. If \( X \) is an Euler \( \mod 2 \) space, then \( \omega_k(\tilde{X}) = \omega_k(\tilde{X}') \) in \( H_k(X; \mathbb{Z}_2) \).

If \( X \) is a manifold, a QR structure is said to be smooth if its second barycentric subdivision is a smooth triangulation.

Corollary 3. If \( \tilde{X} \) is a smooth QR structure on a smooth manifold, then \( \omega_k(\tilde{X}) \) is Poincaré dual to the \( (n-k) \)th Stiefel-Whitney cohomology class.

Proof of theorem. Let \( \mu \) be a \( k \)-cell in \( \tilde{X}' \) represented by \( \langle \sigma_0^{a_0} \sigma_1 \ldots \sigma_k \rangle \). We must show that the number of \( (k+1) \)-cells \( \eta \) of which \( \mu \) is part of the boundary is congruent \( \mod 2 \) to \( \sum_{\xi} [\xi, \mu] = \sum_{\xi} [\tau, \sigma_k] \); i.e., \( \mu \) appears in the boundary of \( C_{k+1}(\tilde{X}) \) if and only if \( \mu \subset \sigma_k \subset X_1 \). To choose such an \( \eta \) is to
make exactly one choice as in steps 1–3 of Theorem 1. Of these, there are exactly $\sum_{k} \lceil \tau, \sigma \rceil \mod 2$, proving the theorem.

**Proof of Corollary 1.** The $C_p(X)$ are all cycles iff $X_1 = \emptyset$.

**Proof of Corollary 2.** Consider the QR structure on $X \times I$, denoted $\overline{X \times I}$, obtained by taking the product of the given structure on $X$ with the obvious one on $I$, and then subdividing $X \times \{1\}$. It is easy to check that if $X$ is $E(2)$, then $\partial(C_{k+1}(X \times I)) = \omega_k(X) \times \{0\} \cup \omega_k(\overline{X}) \times \{1\}$. Thus, these two cycles are homologous. (Note. This result appears to be well known but we have not seen it written down. It is probably due to Toledo or Sullivan.)

**Proof of Corollary 3.** This is well known for triangulations [Halperin and Toledo], and by Corollary 2, the classes are invariant under subdivision. $\square$

2. Applications. In this section, we apply the above ideas to projective spaces, lens spaces and surfaces.

**Proposition 1.**

$$\omega_k(\overline{RP^n}) = \left(\frac{n+1}{n-k}\right)Z_k \pmod{2},$$

where $Z_k$ is the generator of $H_k(RP^n; \mathbb{Z}_2)$, $0 < k < n$.

**Proof.** $RP^n$ denotes $R\mathbb{P}^n$ with the familiar cell structure with one cell $\sigma_i$ in each dimension $i$, $0 < i < n$. $[\sigma_i, \sigma_j] = 2$ for each $i > j$ and we denote the two lifts of $\sigma_i$ to $\sigma_i$ by 0 and 1 – 0 denotes the lift to the upper hemisphere and 1 the composition of 0 with the antipodal map – so that composition of lifts acts like mod 2 addition. The diagram below shows how this scheme works in $RP^2$.

![Diagram of RP^2](image)

For each sequence $i_0 < \cdots < i_k$ of integers from 0 to $n$, $\langle i_0i_1\cdots i_k \rangle$ will denote the chain composed of all $k$-cells in $\overline{RP^n}$ whose representations, ignoring lifts, are $\langle \sigma_{i_0}, \sigma_{i_1}, \cdots, \sigma_{i_k} \rangle$: the same convention will hold if some, but
not all, lifts are specified. The fact that \([σ_i, σ_j]\) is always even implies that \(\langle i_0 i_1 \ldots i_k \rangle\) is a mod 2 cycle. Now \(\langle 01 \ldots k \rangle\) clearly represents \(Z_k\), since, geometrically, it is the standard copy of \(RP^k\) in \(RP^n\). Furthermore, any other chain \(\langle i_0 i_1 \ldots i_k \rangle\) is homologous to \(\langle 01 \ldots k \rangle\) since their mod 2 difference is the boundary of

\[\langle 0^0_0 i_1 \ldots i_k \rangle + \langle 0^1_0 i_1 \ldots i_k \rangle + \langle 0^2_1 i_2 \ldots i_k \rangle + \cdots + \langle 0^1 \ldots i_k \rangle.\]

(If there is a repeat in any of these terms, e.g., \(i_0 = 0\), that term should be ignored.) Since the sum of all the \(\langle i_0 \ldots i_k \rangle\) is the \(k\)-skeleton of \(RP^n\) and there are \(\binom{n+1}{k+1}\) of them,

\[ω_k(RP^n) = \binom{n+1}{k+1}Z_k = \binom{n+1}{n-k}Z_k.\]

Now consider a generalized lens space \(L = L(p; q_1, \ldots, q_n)\) of dimension \(m = 2n - 1\). The CW structure described in [Cohen, p. 90] is a QR structure, denoted \(\overline{L}\). Since \(L\) has no interesting \(Z_2\) homology if \(p\) is odd, we assume \(p\) is even, in which case \(H_k(L, Z_2) \cong Z_2\) for \(0 \leq k \leq m\) with generator \(Z_k\).

**Proposition 2.** \(ω_k(\overline{L}) = \binom{m+1}{m-k}Z_k, 0 \leq k \leq m.\)

**Proof.** The proof generalizes that of Proposition 1. The CW structure has one cell \(σ_i\) in each dimension \(0 \leq i \leq m\) and \([σ_i, σ_j] = p \ (i > j)\) except for \([σ_{2j+1}, σ_{2j}] = 2\). The various lifts can be denoted by \(0, \ldots, p-1\), with the proviso that in the case \(σ_{2j} \subset σ_{2j+1}\), only 0 and 1 are allowed, and composition of lifts acts like mod \(p\) addition. Again \(\langle i_0 \ldots i_k \rangle\) will denote the chain composed of all cells represented, ignoring lifts, by \(\langle σ_{i_0} \ldots σ_{i_k} \rangle\)—this is a cycle because \([σ_i, σ_j]\) is always even. \(\langle 01 \ldots k \rangle\) clearly represents \(Z_k\) since geometrically, it is the \(k\)-skeleton of \(\overline{L}\). But any such chain is homologous to \(\langle 01 \ldots k \rangle\) since mod 2,

\[\langle i_0 i_1 \ldots i_k \rangle - \langle 01 \ldots k \rangle = \partial[\langle 0^0_0 i_1 \ldots i_k \rangle + \langle 0^1_0 i_1 \ldots i_k \rangle + \cdots + \langle 0^1 \ldots i_k \rangle].\]

(Same convention as in Proposition 1 as to repeated entries.) Therefore, \(ω_k(\overline{L}) = \binom{m+1}{m-k}Z_k.\)
For a multiple projective plane $mP^2$, the shaded 2-chain below is an homology between $C_1(mP^2)$ and the dotted 1-chain, which clearly has the property described above.

The remaining case of $S^2$ is left to the reader.

REFERENCES


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