FOUR METRIC CONDITIONS CHARACTERIZING ČECH DIMENSION ZERO

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Abstract. If \((X,d)\) is a metric space let \(d_x(y) = d(x,y)\). It is proved that if each \(x\) in \(A\) has a neighbourhood \(P\) with \(d_x(P)\) not dense in any neighbourhood of 0 in \([0,\infty)\) then \(\text{Ind} \ X = 0\). This metric condition characterizes metrizable spaces which have Čech dimension zero. Three other metric characterizations are given.

Introduction. In [1], [2] and [3] metric conditions sufficient for a metrizable topological space to have Čech dimension zero are given. In [3, Theorem 3.4, p. 66] it is shown that if a space \(X\) has a compatible metric \(d\) with range not dense in any neighbourhood of 0 in \([0,\infty)\) then \(\text{Ind} \ X = 0\). In this paper it is shown that the conclusion is the same when the condition on the metric holds only "locally" or "pointwise". Specifically, if \(d\) satisfies for each \(x\) in \(X\) there is a neighbourhood \(P\) of \(x\) such that \(d_x(P)\) is not dense in any neighbourhood of 0 in \([0,\infty)\) then \(\text{Ind} \ X = 0\). Conversely, if a metrizable space has Čech dimension zero then there exists a compatible metric satisfying the given condition globally and hence locally. Three other related characterizations are given.

First two preliminary constructions are made. Let \((X,d)\) be a metric space. If \(A \subset X\) is not empty let \(d_A(x) = \inf \{d(a,x) : a \in A\}\) for each \(x\) in \(X\). Let \(d_x = d_{\{x\}}\). For each \(n\) in \(\mathbb{N}\) let
\[
E(n) = \{x \in X : d_x(X) \text{ is not dense in } [2^{-n-1},2^{-n}]\}.
\]
Then each \(E(n)\) is open.

For each pair of rational numbers \(r\) and \(s\) with \(0 < r < s\) let
\[
A(r,s) = \{x \in X : d_x(X) \cap [r,s] = \emptyset\}.
\]
For each rational \(\epsilon\) with \(r < \epsilon < s\) let
\[
\mathcal{B}(r,\epsilon,s) = \{B(x,\epsilon) : x \in A(r,s)\}
\]
where \(B(x,\epsilon)\) is the metric ball of center \(x\) and radius \(\epsilon\). The family \(\mathcal{B}(r,\epsilon,s)\) is closure preserving and consists of sets which are closed and open. Well order \(A(r,s)\) and, for each \(x\) in \(A(r,s)\) and \(\epsilon\) in \(Q\) with \(r < \epsilon < s\), let

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Let
\[ D(x, \varepsilon) = B(x, \varepsilon) \setminus \bigcup_{y \prec x} B(y, \varepsilon). \]

The family \( \mathcal{D}(r, \varepsilon, s) = \{ D(x, \varepsilon) : x \in A(r, s) \} \) is closure preserving and consists of disjoint closed and open sets. Finally let
\[ \mathcal{B} = \{ \mathcal{D}(r, \varepsilon, s) : 0 < r < \varepsilon < s \text{ and } r, \varepsilon, s \in \mathbb{Q} \}. \]
Then \( \mathcal{B} \) is a \( \sigma \)-locally finite family of subsets of \( X \) which are closed and open.

**Theorem 1.** Let \( (X, d) \) be such that each \( x \) in \( X \) has a neighbourhood \( P \) satisfying \( d(x, P) \) is not dense in any neighbourhood of 0 in \([0, \infty)\). Then \( \text{Ind} X = 0 \).

**Proof.** Under the given conditions the family \( \mathcal{B} \) is a \( \sigma \)-locally finite base for \( X \) consisting of sets which are closed and open. Thus, by [4, Theorem 5, p. 291], \( \text{Ind} X = 0 \).

The hypothesis of Theorem 1 may be restated as follows: each \( x \) in \( X \) is in \( E(n) \) for an infinite number of integers \( n \) in \( \mathbb{N} \).

**Theorem 2.** Let \( E(n) \) be dense in \( X \) for an infinite number of positive integers \( n \). Then \( \text{Ind} X = 0 \).

**Proof.** The proof consists in showing that \( \mathcal{B} \) is a base for \( X \). Let \( P \) be open and let \( x \in P \). Let \( n \) in \( \mathbb{N} \) satisfy \( B(x, 2^{-n+1}) \subset P \) and \( E(n) \) is dense in \( X \). Let \( y \in E(n) \cap B(x, 2^{-n-1}) \). Then there exist rational numbers \( r, s, \varepsilon \), with \( 2^{-n-1} < r < \varepsilon < s < 2^{-n} \) and such that \( y \in A(r, s) \). Because \( x \in B(y, \varepsilon) \) it follows that there is a \( z \) in \( A(r, s) \) with \( x \in D(z, \varepsilon) \subset B(y, \varepsilon) \subset B(x, 2^{-n+1}) \subset P \). This completes the proof.

**Theorem 3.** If \( d_n(X) \) is not a neighbourhood of 0 in \([0, \infty)\) for all closed subsets \( F \) in \( X \) then \( \text{Ind} X = 0 \).

**Proof.** Let \( P \subset X \) be any open subset. Then there exists a sequence \( a_n \downarrow 0 \) in \([0, \infty)\) such that \( a_n \not\in d_{X-P}(X) \) for each \( n \) in \( \mathbb{N} \).

Let \( G_1 = \{ x \in X : d_{X-P}(x) > a_i \} \). Then \( P = G_1 \cup (G_2 \setminus G_1) \cup (G_3 \setminus G_2) \cup \ldots \). The sets in this union are closed, open and disjoint. Since every open set has such a decomposition it follows from [4, Theorem 5, p. 291] that \( \text{Ind} X = 0 \).

**Theorem 4.** Let \( (X, d) \) be a metric space and suppose that there exists a sequence \( (d_n) \) of continuous metrics on \( X \) converging uniformly to \( d \) and such that each \( d_n \) satisfies either the hypothesis of Theorem 1 or of Theorem 2. Then \( \text{Ind} X = 0 \).

**Proof.** Let \( X_n = (X, d_n) \) and consider the mapping
defined by $x \rightarrow (x,x,x,\ldots)$. The map $f$ is an embedding. By Theorems 1 and 2, for each $n$, $\text{Ind } X^n = 0$. Hence $\text{Ind } Y = 0$. Therefore $\text{Ind } X = 0$ since $Y$ is metrizable. This completes the proof.

REFERENCES