THE DOMAIN COVERED BY A TYPICALLY-REAL FUNCTION

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Abstract. We find the largest possible domain that is covered by \( f(E) \) for every typically-real function \( f(z) \). In the process we obtain a set of universal typically-real functions.

1. A set of universal typically-real functions. We recall that a function

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \]

regular in \( E: |z| < 1 \), is said to be typically-real in \( E \) if \( f(z) \) is real for real \( z \) and only for real \( z \) in \( E \) [5]. We let \( TR \) denote the set of all functions with the normalization (1) that are typically-real in \( E \).

We first describe a particularly interesting universal typically-real function.

**Theorem 1.** The function

\[ G(z) = \tan \left( \frac{\pi z}{1 + z^2} \right) \]

is in \( TR \). In \( E \), the function \( G(z) \) assumes each real value exactly once, it omits \( \pm i/\pi \), and it assumes every other complex value infinitely often. Further \( G'(z) \) is never zero in \( E \).

**Proof.** To visualize the behavior of \( G(z) \) we describe the surface \( G(E) \). It consists of two infinite sets of half-planes, \( \text{Im } w > 0 \) and \( \text{Im } w < 0 \), each set joined by a branch point similar in nature to \( \ln(w \pm i/\pi) \). One half-plane of the first set is joined to one half-plane of the second set along the real axis. For each of the other half-planes, the real axis is the boundary.

For simplicity, set \( F(z) = \pi G(z) \) and let \( \zeta = z/(1 + z^2) \). Then

\[ F(z) = \tan \pi \zeta = \frac{\sin \pi \zeta}{\cos \pi \zeta} = \frac{1}{i} \frac{e^{\pi i \zeta} - e^{-\pi i \zeta}}{e^{\pi i \zeta} + e^{-\pi i \zeta}} = i \frac{1 - e^{2\pi i \zeta}}{1 + e^{2\pi i \zeta}}. \]

Consider first the transformation \( \zeta = z/(1 + z^2) \) on the upper half of the unit disk; \( E^+ \). It is easy to see that \( \zeta(E^+) \) is the half-plane \( \text{Im } w > 0 \). The interval \([-1, 1]\) goes into the interval \([-1/2, 1/2]\), and the upper boundary of \( E^+ \) goes into the remainder of the real axis in a one-to-one manner.

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\(^1\)H(z) is a universal typically-real function (with respect to \( w = a + bi \)) if every typically-real function that omits \( a + bi \) is subordinate to \( H(z) \).

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Next $\eta = e^{2\pi i t}$ takes this half-plane onto the interior of infinitely many copies of the unit disk, except for the point $\eta = 0$ where the infinitely many copies have a branch point of type $\ln \eta$. In this transformation the interval $-1/2 < \xi < 1/2$ goes into the boundary of the unit disk with $\eta(-1/2) = \eta(1/2) = -1$. Finally $F(z) = i(1 - \eta)/(1 + \eta)$ carries this infinite family of disks onto an infinite family of half-planes, $\text{Im } w > 0$, tied together at $w = i$. The composition of these mappings carries the interval $-1 < z < 1$ onto the real axis on one of the infinitely many half-planes. The Schwarz-reflection principle shows that $F(z)$ carries the full disk $E$ onto a surface of the type described at the beginning of the proof. It is then obvious geometrically that $G(z) \equiv F(z)/\pi$ has all the properties described in Theorem 1. It is easy to give an analytic proof that in $E$, $G(z) \neq \pm i/\pi$, $G'(z) \neq 0$, and $G(z)$ assumes every other value $a + bi$ ($b \neq 0$) infinitely often.

The function $G(z)$ can be regarded as a universal function because the surface $G(E)$ will carry $f(E)$ for any other typically-real function that omits $\pm i/\pi$. If $b \neq 0$, then an easy transformation of $F(z)$ (see the next section) gives a universal function for typically-real functions that omit $z = a \pm bi$.

2. The region covered by $f(z)$. Brannan and Kirwan [1], proved that for every $f(z)$ in $TR$, the domain $f(E)$ covers the disk $|z| < 1/4$. We will determine the maximum domain $D$ covered by $f(E)$. It turns out that $D$ is not a disk, but of course it contains the disk $|z| < 1/4$.

We begin with the simple case in which $f(z)$ omits $bi$ in $E$, where $b > 0$. Throughout our discussion we use the fact that $f(E)$ is symmetric with respect to the real axis for any typically-real function. Hence we can restrict our attention to the upper half-plane $\text{Im } w > 0$.

Let $f(z) \in TR$ and omit $bi$, where $b > 0$. Then $f(z)/\pi b$ omits $i/\pi$. If $G^{-1}(w)$ denotes the inverse function of $G(z)$, defined on the surface $G(E)$, then
\begin{equation}
B(z) \equiv G^{-1}(f(z)/\pi b)
\end{equation}
is regular in $E$ and satisfies the conditions of Schwarz's Lemma. Hence $f(z) = \pi b G(B(z))$, \begin{equation}
f'(z) = \pi b G'(B(z)) B'(z)
\end{equation}
and at $z = 0$, we have
\begin{equation}
1 = f'(0) = \pi b G'(0) B'(0) < \pi b,
\end{equation}
since $G'(0) = 1$ and $0 < B'(0) < 1$. Further, equality can occur if and only if $B(z) \equiv z$ and $f(z) \equiv G(z)$. Thus if $f(z) \in TR$ and omits $bi$ with $b > 0$ then $b > 1/\pi$ and the inequality is sharp.

Now assume that $f(z) \in TR$ and omits $a + bi$ where $a \neq 0$ and $b > 0$. For convenience set $a + bi = \rho e^{i\alpha}$ where $0 < \alpha < \pi$ and $\alpha \neq \pi/2$ since $a \neq 0$. We let $c$ be the unique real root in $E$ of the equation
\begin{equation}
G(z) = -\pi^{-1} \cot \alpha = -\pi^{-1}ab^{-1},
\end{equation}
where \( G(z) \) is our universal function (2). Theorem 1 assures us that there is exactly one root in \((-1, 1)\). A brief computation using \( \tan(\pi z/(1 + z^2)) = \tan(\alpha - \pi/2) \) gives

(8) \[ z^2 + (2\pi/(\pi - 2\alpha))z + 1 = 0, \]

and hence

(9) \[ c = (-\pi + 2\sqrt{\pi\alpha - \alpha^2})/(\pi - 2\alpha). \]

With this \( c \) set

(10) \[ H(z) = \pi b G((z + c)/(1 + cz)) + a. \]

Then the function \( H(z) \) is a universal typically-real function that omits \( a + bi \), and \( H(0) = 0 \). If \( H^{-1}(w) \) is the inverse function of \( H(z) \) defined on the surface \( H(E) \), then \( H^{-1}(f(z)) = B(z) \), where \( B(z) \) satisfies the conditions of Schwarz's Lemma. Consequently \( f(z) = H(B(z)) \) and

\[
1 = f'(0) = H'(0)B'(0) = \pi b G'(c)(1 - c^2)B'(0)
\]

(11) \[ 1 = \pi b \csc^2 \alpha \left( \frac{1 - c^2}{1 + c^2} \right)^2 B'(0). \]

Since \( c \) is a root of equation (8) we have

(12) \[ \left( \frac{1 - c^2}{1 + c^2} \right)^2 = \frac{(1 + c^2)^2 - 4c^2}{(1 + c^2)^2} = \frac{[2\pi/(\pi - 2\alpha)]^2 - 4}{[2\pi/(\pi - 2\alpha)]^2} = \frac{4}{\pi^2}(\pi\alpha - \alpha^2). \]

We use this in (11), together with \( 0 < B'(0) < 1 \), and \( b \csc \alpha = \rho \). This gives

(13) \[ \rho > \frac{\pi}{4} \frac{\sin \alpha}{\alpha(\pi - \alpha)}, \quad 0 < \alpha < \pi, \alpha \neq \pi/2, \]

with equality if and only if \( B(z) \equiv z \). But equality is possible. When \( B(z) \equiv z \), then \( f(z) = H(z) \).

We note that in the proof we were forced to assume that \( a \neq 0 \), \( a \neq \pi/2 \). But if we set \( \alpha = \pi/2 \), equation (13) gives \( \rho \equiv b > 1/\pi \). Further as \( \alpha \to 0^+ \) or \( \alpha \to \pi^- \) we have

(14) \[ \lim_{\alpha \to 0^+} \frac{\sin \alpha}{\alpha(\pi - \alpha)} = \frac{1}{4}, \]

the Koebe constant. That \(|a| > 1/4 \) can be established by the same type of subordination argument, but it is interesting to observe that in this special case \((b = 0)\) the universal functions are \( F(z) = z/(1 \pm z)^2 \), a rather abrupt change from \( G(z) \) given by (2). We have proved
Theorem 2. Let $D$ be the largest domain such that $D \subset f(E)$ for every $f(z)$ in $TR$. Then the upper boundary of $D$ is given in polar coordinates $(\rho, \theta)$ by

$$\rho = \frac{\pi \sin \theta}{4\theta (\pi - \theta)},$$

and the lower boundary is obtained by reflecting this curve in the real axis. The points $w = \pm 1/4$ are also part of the boundary of $D$.

3. Comments. M. S. Robertson [4] has proved that if $f(z) \in TR$, then it has an integral representation

$$f(z) = \frac{1}{\pi} \int_0^\pi \frac{z}{1 - 2z \cos t + z^2} \, d\mu(t),$$

where $\mu(t)$ is nondecreasing on $[0, \pi]$ and

$$\int_0^\pi d\mu(t) = \pi.$$

Since $G(z)$ has such unusual properties it would be interesting to have $\mu(t)$ explicitly for this function. For one thing, we could compute the coefficients in the power series for $G(z)$—coefficients that seem to be difficult to compute directly from (2).

It is known [5] that if $f(z)$ is a normalized typically-real function, then

$$g(z) = \left[ (1 - z^2)/z \right] f(z) = 1 + p_1 z + p_2 z^2 + \ldots$$

is in $P$, the set of normalized functions with $\text{Re} \ g(z) > 0$ in $E$. One might ask for the largest domain covered by every function in this set. Here the corresponding question is trivial because for the set $P$, the set $D$ that is always covered consists of the single point $w = 1$ (and does not form a domain). Even if we consider the subset of univalent functions in $P$, we get the same trivial answer: $D = \{1\}$.

The largest domain $D$ covered by $f(E)$ for every $f(z)$ in a certain set $M$ of normalized functions has been treated by other writers. Classical distortion theorems give the disk $|w| < 1/4$ if $M$ is the set $S$ of normalized univalent functions, and $|w| < 1/2$ for the subset of convex functions. Goodman and Saff [2] found $D$ for the subset of $S$ of functions convex in the direction of the imaginary axis. Further McGregor [3] has determined $D$ for subsets of $S$ for which all the coefficients are real and: (a) $f(z)$ is starlike, (b) $f(z)$ is convex, and (c) $f(z)$ is convex in the direction of the imaginary axis.

The largest domain covered by $f(E)$ for every $f(z)$ in $M$ is sometimes referred to as the Koebe domain for the set $M$. For more references to papers on Koebe domains the reader should consult Bernardi's Bibliography of schlicht functions, Part I (1966) and Part II (1977), Courant Institute of Mathematical Sciences, New York University, New York.

**References**


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