THIRD ORDER LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

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Abstract. The third order linear differential equation $Ly = y''' + p_2(t)y'' + p_1(t)y' + p_0(t)y = 0$ is studied, where $p_i(t)$ are continuous real-valued and periodic of period $\omega > 0$. Various criteria are obtained which guarantee "partial" asymptotic stability or instability by means of effective bounds on the Floquet characteristic multipliers of $Ly = 0$.

1. Introduction. Consider the third order differential equation

$Ly = y''' + p_2(t)y'' + p_1(t)y' + p_0(t)y = 0$

where the $p_i(t)$ are continuous on $(-\infty, +\infty)$ and periodic of period $\omega > 0$. The stability properties of (1.1) are determined by the characteristic multipliers; i.e., by the roots of the Floquet characteristic equation

$\lambda^3 - A_2\lambda^2 + A_1\lambda - B = 0$

where $A_1, A_2$ are given in terms of the fundamental system of solutions of (1)

$y_0, y_1, y_2$, with $y_i^{(j)}(0) = \delta_{ij}, 0 < i, j < 2$, as follows:

$A_2 = y_0(\omega) + y_1(\omega) + y_2(\omega),$

$A_1 = \begin{vmatrix} y_0(\omega) & y_1(\omega) & y_2(\omega) \\ y_0(\omega) & y_1(\omega) & y_2(\omega) \\ y_0(\omega) & y_1(\omega) & y_2(\omega) \end{vmatrix} + \begin{vmatrix} y_0(\omega) & y_1(\omega) & y_2(\omega) \\ y_0(\omega) & y_1(\omega) & y_2(\omega) \\ y_0(\omega) & y_1(\omega) & y_2(\omega) \end{vmatrix} + \begin{vmatrix} y_0(\omega) & y_1(\omega) & y_2(\omega) \\ y_0(\omega) & y_1(\omega) & y_2(\omega) \\ y_0(\omega) & y_1(\omega) & y_2(\omega) \end{vmatrix},$

and $B = e^{-\omega}, J = (1/\omega)\int_0^\omega p_2(t) dt$. In [12] stability of (1.1) is related to certain periodic and multi-point de la Valée Poussin type boundary value problems and various necessary and/or sufficient conditions are obtained for stability and instability of (1.1). These problems were also considered in [2] via the method of lower and upper solutions for the corresponding second order Riccati equation

$u'' + f(t, u, u') = 0, \quad u = y'/y,$

where

$f(t, u, u') = 3uu' + p_2u' + u^3 + p_2u^2 + p_1u + p_0.$

In this paper we shall be interested in obtaining lower and/or upper bounds on some or all of the characteristic multipliers which will be denoted

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by $\lambda_1, \lambda_2, \lambda_3$. Since $B = \lambda_1\lambda_2\lambda_3$ it is clear that a necessary condition for asymptotic stability (i.e., $|\lambda_i| < 1$, $i = 1, 2, 3$) is that $J > 0$. We shall be interested in finding sufficient conditions for "partial" asymptotic stability or instability. That is, we obtain conditions under which the asymptotically stable, stable, and unstable manifolds, denoted by $\mathcal{S}$, $\mathcal{S}$, and $\mathcal{T}$, respectively (i.e., the subspace of the solution space of (1.1) corresponding to characteristic multipliers of modulus $|\lambda| < 1$, $|\lambda| = 1$, $|\lambda| > 1$), have dimension $k$, $0 < k < 3$. We set $d_u = \dim \mathcal{S}$, $d_s = \dim \mathcal{S}$, and $d_u = \dim \mathcal{T}$. A solution of (1.1) will be said to be oscillatory if it has infinitely many zeros in a neighborhood of $+\infty$. The set of all oscillatory solutions will be denoted by $\mathcal{O}$. The set of all solutions of (1.1) which never vanish on $(-\infty, +\infty)$ will be denoted by $\mathcal{R}$.

If for each $0 < t < \omega$ the characteristic polynomial of (1.1),

$$
\sigma(t, \rho) = \rho^3 + p_2(t)\rho^2 + p_1(t)\rho + p_0(t),
$$

has real roots $\rho_i(t)$, $i = 1, 2, 3$, which are separated by constants (i.e., $\rho_i(t) < \mu_1 < \rho_2(t) < \mu_2 < \rho_3(t)$, $0 < t < \omega$), then (1.1) is disconjugate on $(-\infty, +\infty)$; that is, no nontrivial solution of (1.1) has more than two zeros on $(-\infty, +\infty)$. (See e.g., [1].) In this case, $\mu_1$ and $\mu_2$ are lower and upper solutions, respectively, of the Riccati equation (1.3) and the stability properties of (1.1) may be related to the sign of $\mu_1, \mu_2$ ([2]). We are interested here in obtaining criteria for the case when (1.1) has (perhaps) oscillatory solutions.

Consider the sets $S_i$, $T_i$, $i = 1, 2$, defined by

$$
S_1 = \{\rho: \sigma(t, \mu) < 0 \text{ for all } \mu < \rho, t \in [0, \omega]\},
$$

$$
S_2 = \{\rho: \sigma(t, \rho) < 0, t \in [0, \omega]\},
$$

$$
T_1 = \{\rho: \sigma(t, \mu) > 0 \text{ for all } \mu > \rho, t \in [0, \omega]\},
$$

$$
T_2 = \{\rho: \sigma(t, \rho) > 0, t \in [0, \omega]\}.
$$

Let $\alpha_i = \sup S_i$, $\beta_i = \inf T_i$, $i = 1, 2$. Thus, $S_1 \subseteq S_2$, $T_1 \supseteq T_2$ and $\alpha_1 < \alpha_2$, $\beta_2 < \beta_1$ and $\sigma(t, \alpha_i) < 0 < \sigma(t, \beta_i)$, $i = 1, 2$. We shall see below that the stability properties of (1.1) can be related to the sets $S_i$, $T_i$, and that it is often possible to obtain effective lower bounds on $\alpha_i, \alpha_2$ and upper bounds on $\beta_1, \beta_2$ in terms of the coefficients $p_i(t)$. The main results are in §2 below and in §3 we briefly discuss their applicability.

2. We begin this section with a result which yields upper and lower bounds on the set of positive characteristic multipliers of (1.1). Since the coefficients in equation (1.2) are all real and $B = \lambda_1\lambda_2\lambda_3 > 0$, there always exists at least one positive characteristic multiplier.

**Theorem 2.1.** Let $\alpha_1, \beta_1$ be defined as above (i.e., $\alpha_1 = \sup S_1$, $\beta_1 = \inf T_1$) and assume that for each fixed $\rho \in S_1 \cup T_1$, $\sigma(t, \rho) \equiv 0$, for $t \in [0, \omega]$. Assume further that for each $0 < \tau < \omega$ equation (1.1) is disconjugate on $[\tau, \tau + \omega]$. Then any positive characteristic multiplier $\lambda$ of (1.1) satisfies
\( e^{\alpha \omega} < \lambda < e^{\beta \omega}, \)

and the corresponding solution \( y_\lambda \) of (1.1) belongs to \( \mathcal{R} \).

**Proof.** From Floquet theory, if \( \lambda > 0 \) is a multiplier of (1.1), then the corresponding solution of (1.1) may be written \( y_\lambda(t) = \phi(t)e^{\rho t} \), where \( \rho = \ln \lambda/\omega, \phi(t+\omega) = \phi(t) \) for all \( t \), and \( \phi(t) > 0 \) on \([0, \omega]\) by disconjugacy. Thus, \( y_\lambda \in \mathcal{R} \). To prove (2.1) (i.e., \( \alpha_1 < \rho < \beta_1 \)), suppose first that \( \rho > \beta_1 \). Then with \( z(t) = e^{\rho t} \) we find that \( Lz(t) = e^{\rho t} \sigma(t, \rho) > 0 \). Let

\[
  k = \min_{0 < t < \omega} \phi(t) = \phi(t_0)
\]

and let \( v(t) = y_\lambda(t) - k z(t) \). Thus, \( v(t) > 0 \) on \([t_0, t_0 + \omega]\) and \( v(t) = v'(t_0) = v(t_0 + \omega) = v'(t_0 + \omega) = 0 \). Since \( Ly = 0 \) is disconjugate on \([t_0, t_0 + \omega]\), the Green’s function \( G_{21}(t, s) \) for the boundary value problem

\[
  L y = f, \quad y(t_0) = y'(t_0) = y(t_0 + \omega) = 0
\]

exists and is nonnegative for \( t, s \in [t_0, t_0 + \omega] \). Furthermore, the solution of (2.2) may be written as

\[
  y(t) = \int_{t_0}^{t_0 + \omega} G_{21}(t, s)f(s) \, ds.
\]

Since \( Lv(t) = -k e^{\rho t} \sigma(t, \rho) = f(t) < 0 \) on \([t_0, t_0 + \omega]\), we have

\[
  v(t) = \int_{t_0}^{t_0 + \omega} G_{21}(t, s)f(s) \, ds < 0
\]

which implies \( v(t) \equiv 0 \), i.e., \( \sigma(t, \rho) \equiv 0 \), a contradiction to our assumption. Therefore, \( \rho < \beta_1 \). Similarly, with

\[
  v_1(t) = k_1 z(t) - y(t), \quad k_1 = \max_{0 < t < \omega} \phi(t) = \phi(t_1),
\]

we may show that \( \rho > \alpha_1 \). This completes the proof.

As an immediate consequence of Theorem 2.1, we have

**Corollary 2.2.** In addition to the hypotheses of Theorem 2.1 assume \( \beta_1 < 0 \) \((\alpha_1 > 0)\). Then any positive characteristic multiplier \( \lambda \) of (1.1) satisfies \( \lambda < 1 \), and \( \sigma_* > 1 \), \( \mathcal{R} \cap \mathcal{S} \neq \emptyset \) \((\lambda > 1, \sigma_* > 1 \text{ and } \mathcal{R} \cap \mathcal{S} \neq \emptyset)\).

We now begin a more detailed analysis of \( \mathcal{R} \), \( \mathcal{S} \), and \( \mathcal{H} \). It is convenient to consider three cases: (i) \( J > 0 \), (ii) \( J = 0 \), (iii) \( J < 0 \). Since \( B = e^{-\omega J} = \lambda_1 \lambda_2 \lambda_3 \), it follows that \( d = 3 \) (i.e., asymptotic stability) occurs only in case (i), \( d = 3 \) (stability) occurs only in case (ii), and \( d > 1 \) (instability) always occurs in case (iii). Our primary concern is to obtain criteria under which \( d, d_2, d_* \) take on various values of \( k, 0 < k < 3 \). In what follows, we will occasionally assume that the following condition holds (see also [12]):

**Condition A.** We say that equation (1.1) satisfies Condition A in case for any \( 0 < \tau < \omega \) the boundary value problem

\[
  Ly = 0, \quad y(\tau) = y(\tau + \omega) = y(\tau + 2\omega) = 0
\]

has a Green’s function.
Clearly, Condition A will hold if, for example, equation (1.1) is disconjugate on \([\tau, \tau + 2\omega]\) for all \(0 < \tau < \omega\). However, because of the particular properties of the three-point problem Condition A may also hold even though solutions of (1.1) exist with three zeros on \([\tau, \tau + 2\omega]\). Sufficient conditions for A to hold may be found in [12].

We begin with Case (i), \(J > 0\):

**Theorem 2.3.** Let \(J > 0\) and assume (1.1) is disconjugate on \([\tau, \tau + \omega]\) for all \(0 < \tau < \omega\).

(a) If \(-J < \alpha_1 < \beta_1 < 0\) (i.e., \((-\infty, -J) \subseteq S_1, [0, +\infty) \subseteq T_1\)) and if for each \(\rho \in (-\infty, -J) \cup [0, +\infty), \sigma(t, \rho) \equiv 0\) on \([0, \omega]\), then \(d_a > 2, d_s = 0, \text{ and } \emptyset \cap \mathcal{R} \neq \emptyset\). Further, if Condition A holds, then \(d_a = 3\).

(b) If \([-J, 0] \subseteq S_2 \cup T_2\) and \(\sigma(t, \rho) \equiv 0\) on \([0, \omega]\) for each \(\rho \in [-J, 0]\), then \(d_a > 1, d_a = 1, d_s = 0, \text{ and } \emptyset \cap \mathcal{R} \neq \emptyset\).

(c) If Condition A holds, if \((-\infty, 0] \subseteq S_1, [-J, +\infty) \subseteq T_1\) and \(\sigma(t, \rho) \equiv 0\) on \([0, \omega]\) for \(\rho < 0\) (\(\rho > -J\)), \(A_2 X_3 = B/X_1 < 1\) so that one of \(A, X_2, X_3\) satisfies \(|A, X_2, X_3| < 1\). To be specific, assume \(|A, X_2, X_3| < 1\). Thus, \(d_a > 2\) and \(d_s = 0\) since complex multipliers occur only in conjugate pairs. This proves the first part of (a). Now if Condition A holds, then there are no negative multipliers. For if \(\lambda < 0\) is a multiplier, then there exists a solution \(y(t) \equiv 0\) of (1.1) with \(y(t + \omega) = \lambda y(t)\) and hence \(y(\tau) = 0\) for some \(0 < \tau < \omega\). Thus, \(y(\tau) = y(\tau + \omega) = y(\tau + 2\omega) = 0\), a contradiction. We conclude, therefore, that either all multipliers are real and satisfy \(B < \lambda < 1\), \(i = 1, 2, 3\), or \(\lambda_2, \lambda_3\) are complex with \(|\lambda_2| = |\lambda_3| < 1\), and in either case \(d_a = 3\).

(b) If \([-J, 0] \subseteq S_2 \cup T_2\), then since \(\sigma(t, \rho) \equiv 0\) for \(\rho \in [-J, 0]\), it follows that \(S_2 \cap T_2 = \emptyset\) so that \([-J, 0] \subseteq S_2\) or \([-J, 0] \subseteq T_2\). To be specific, assume \([-J, 0] \subseteq S_2\). If there exists a multiplier \(\lambda \in [B, 1]\), then by Floquet theory \(y_{\lambda}(t) = (\phi(t)) e^{\rho t}, \rho = \ln \lambda/\omega \in [-J, 0]\), and as in Theorem 2.1 \(\phi(t) > 0\) on \([0, \omega]\) with \(k = \max_{0 < t < \omega} \phi(t) = \phi(t_0)\) and \(v(t) = ke^{\rho t}, y_{\lambda}(t) > 0\) on \([0, \omega]\) we have \(L v(t) = k \sigma(t), \rho < 0\) and thus, as in Theorem 2.1, \(v(t) \equiv 0 \Rightarrow \sigma(t, \rho) \equiv 0\), a contradiction. If \([-J, 0] \subseteq T_2\), then we use \(v(t) = y_{\lambda}(t) - k_1 e^{\rho t}\), where \(k_1 = \min_{0 < t < \omega} \phi(t) = \phi(t_1)\), and \(L_3 v(t) = -k_1 \sigma(t, \rho) < 0 \Rightarrow v(t) \equiv 0\), a contradiction. It follows that no real multipliers \(\lambda\) satisfy \(B < \lambda < 1\). Therefore, \(0 < \lambda_1 < B\) or \(\lambda_1 > 1\) so that \(B/\lambda_1 = \lambda_2 \lambda_3 > 1\) or \(l < 1\). Thus, \(d_a > 1, d_a > 1, \text{ and } d_s = 0\).

(c) If \((-\infty, 0] \subseteq S_1\), then this implies that there are no positive multipliers.
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\[ \lambda_1 > 1 > B \text{ so that } 0 < \lambda_2 \lambda_3 < 1 \text{ and therefore } \lambda_2, \lambda_3 \text{ must be complex with } |\lambda_2| = |\lambda_3| < 1. \] 

Thus, the corresponding solutions are oscillatory. Similarly, if \([-J, +\infty) \subseteq T_1\), there are no positive multipliers \( B \text{ so that } 0 < \lambda_1 < 1 \) and hence \( \lambda_2 \lambda_3 > 1 \). Thus, \( \lambda_2, \lambda_3 \) are complex. This completes the proof.

A similar result may be obtained in the case \( J = 0 \).

**Theorem 2.4.** Let \( J = 0 \) and assume (1.1) is disconjugate on \([\tau, \tau + \omega]\) for all \( 0 < \tau < \omega \).

(a) If \( [0, +\infty) \subseteq T_1 \), \( \sigma(t, \rho) \equiv 0 \) for each \( \rho \in [0, +\infty) \), and if Condition A holds, then \( d_u = 2, d_a = 1 \), \( \mathcal{U} \) has a basis of solutions \( u_1, u_2 \in \Theta \) and \( \Theta \cap \mathcal{U} \neq \emptyset \).

(b) If \( (-\infty, 0] \subseteq S_1 \), \( \sigma(t, \rho) \equiv 0 \) for each \( \rho \in (-\infty, 0] \), and if Condition A holds, then \( d_u = 1, d_a = 2 \), \( \Theta \) has a basis of solutions \( y_1, y_2 \in \Theta \) and \( \Theta \cap \mathcal{U} \neq \emptyset \).

(c) If \( \sigma(t, 0) = p_0(t) \equiv 0 \) and \( 0 \in S_2 \) or \( 0 \in T_2 \), then \( d_u > 1, d_a > 1 \), and \( d_d = 0 \).

**Proof.** (a) If \( [0, +\infty) \subseteq T_1 \), Theorem 2.1 implies \( 1 < \lambda_1 < B = \lambda_1 \lambda_2 \lambda_3 \). Hence, \( \lambda_2 \lambda_3 > 1 \). Condition A implies that there are no negative multipliers and since there are no real multipliers \( \geq 1 \), it follows that \( \lambda_2, \lambda_3 \) are complex with \( |\lambda_2| = |\lambda_3| > 1 \), and the corresponding solutions of (1.1) are oscillatory.

(b) If \( (-\infty, 0] \subseteq S \), then by Theorem 2.1, \( 1 < \lambda_1 \) so that \( \lambda_2, \lambda_3 \) are complex with \( |\lambda_2| = |\lambda_3| < 1 \).

(c) The hypotheses imply, as in Theorem 2.3(b), that there are no real multipliers \( \lambda = 1 \). Hence, \( 1 < \lambda_1 \Rightarrow \lambda_2 \lambda_3 < 1 \Rightarrow |\lambda_2| < 1 \) or \( 0 < \lambda_1 < 1 \Rightarrow |\lambda_3| > 1 \). Therefore, \( d_u > 1, d_a > 1, \) and \( d_d = 0 \).

For the case \( J < 0 \) for which \( d_d > 1 \) always holds, we have the following result which gives criteria under which \( d_u > 2, d_d = 3 \), or \( d_d > 1 \). This result is analogous to Theorem 2.3.

**Theorem 2.5.** Let \( J < 0 \) and assume (1.1) is disconjugate on \([\tau, \tau + \omega]\) for all \( 0 < \tau < \omega \).

(a) If \( (-\infty, 0] \subseteq S_1 \), \( [-J, +\infty) \subseteq T_1 \) (i.e., \( 0 < \alpha_1 < \beta_1 < -J \)) and if for each \( \rho \in (-\infty, 0] \cup [-J, +\infty) \), \( \sigma(t, \rho) \equiv 0 \) on \([0, \omega]\), then \( d_u > 2, d_a = 0, \) and \( \mathcal{U} \cap \mathcal{N} \neq \emptyset \). Further, if Condition A holds, then \( d_u = 3 \).

(b) If \( [0, -J] \subseteq S_2 \cup T_2 \) and \( \sigma(t, \rho) \equiv 0 \) on \([0, \omega]\) for each \( \rho \in [0, -J] \), then \( d_u > 1, d_d > 1, d_d = 0 \) and \( \mathcal{U} \cap \mathcal{N} \neq \emptyset \).

(c) If Condition A holds, if \( (-\infty, -J) \subseteq S_1 \) \(([0, +\infty) \subseteq T_1) \) and \( \sigma(t, \rho) \equiv 0 \) for \( \rho < -J \) \((\rho > 0) \), then \( d_u = 1, d_u = 2 \), \( \mathcal{U} = \langle u_1 \rangle, u_1 \in \mathcal{N}, \) \( \Theta = \langle u_2, u_3 \rangle, u_2, u_3 \in \Theta \) \((d_u = 2, d_u = 1, \mathcal{U} = \langle u_1, u_2 \rangle, u_1, u_2 \in \Theta, \mathcal{N} = \langle u_3 \rangle, u_3 \in \mathcal{N}) \).

**Proof.** (a) Theorem 2.1 implies \( 1 < \lambda_1 < B \) and \( \mathcal{U} \cap \mathcal{N} \neq \emptyset \). Hence one of the remaining two multipliers, say \( \lambda_2 \), satisfies \( |\lambda_2| > 1 \). If Condition A holds, then there are no negative multipliers and therefore no real multipliers.
This implies $|\lambda_i| > 1$, $i = 1, 2, 3$, so that $d_u = 3$.

(b) If $[0, -J] \subseteq S_2$ (or $\subseteq T_2$), then as in Theorem 2.3(b), no real multiplier satisfies $1 < \lambda < B$. Therefore, either $0 < \lambda_1 < 1$ or $\lambda_1 > B$ and in either case we have $d_u > 1$, $d_s > 1$, $d_s = 0$, and $\mathcal{R} \neq \emptyset$.

(c) The proof is very similar to the proof of part (c) of Theorem 3.

In general, one cannot expect that $\beta_2 < \alpha_2$ holds, for the simple reason that this is sufficient to imply that $L_y = 0$ is disconjugate on $(-\infty, +\infty)$ and consequently all multipliers are real and positive. (Disconjugacy of $L_y = 0$ follows since $\beta_2, \alpha_2$ are lower and upper solutions, respectively, for the Riccati equation (1.3); see [5], [3].) Thus, for example, $\alpha_1 < \beta_2 < \alpha_2 < \beta_1 < 0$ implies that $L_y = 0$ is disconjugate and hence there are no complex or negative multipliers; furthermore, no real multipliers are $> 1$ if $\sigma(t, \rho) \equiv 0$ for $\rho > 0$. This means $L_y = 0$ is asymptotically stable ($d_u = 3$) and, therefore, $J > 0$. Similarly $0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_1$ and $\sigma(t, \rho) \equiv 0$ for $\rho < 0$ implies $d_u = 3$ and $J < 0$. Conditions which are sufficient to guarantee $d_s = 3$ for the case $J = 0$ are, in general, such that a small perturbation of the coefficients $p_i(t)$ leads to instability (i.e., the stability problem is not well-posed; see [10]). The next result yields conditions under which $d_u = d_s = d_v = 1$ in the case $J = 0$.

**Theorem 2.6.** Let $J = 0$ and assume $p_2(t), p_0(t)$ are odd and $p_1(t)$ is even. Assume further that $\beta_2 < \alpha_2$. Then $0 < \lambda_2 < 1 = \lambda_1 < \lambda_3$ (i.e., $d_s = d_u = 1$), and (1.1) is disconjugate on $(-\infty, +\infty)$.

**Proof.** The remarks preceding the theorem imply disconjugacy of (1.1) on $(-\infty, +\infty)$ and hence that all multipliers are real and positive. It suffices, therefore, to show $\lambda_1 = 1$. But this is immediate since by the hypotheses, equation (1.1) is not changed by the transformation $t \rightarrow -t$ and therefore the coefficients $A_1, A_2$ in equation (1.2) satisfy $A_1 = A_2$. Since $B = 1$, it follows that $\lambda_1 = 1$ is a root of (1.2).

As our final result in this section, we have the following theorem which yields $d_u > 1$ with the assumption $p_2(t) \equiv 0$, by an appeal to a result on asymptotic behavior of nonoscillatory solutions for the case when oscillatory solutions exist. (See [6].)

**Theorem 2.7.** Assume $p_0(t) > 0, p_1(t) \leq 0, p_2(t) \equiv 0$, and that (1.1) is disconjugate on $[\tau, \tau + \omega]$ for all $0 < \tau < \omega$. Suppose also that (1.1) has a (nontrivial) solution $u \in \mathcal{D}$. Then

(a) any positive characteristic multiplier satisfies $0 < \lambda < 1$; $d_u > 1$, $d_v > 1$ and (1.1) has three linearly independent solutions $y_1, y_2, y_3$ with $y_1 \in \mathcal{D} \cap \mathcal{R}, y_2, y_3 \in \mathcal{D}$;

(b) if, in addition, Condition A holds, then $d_u = 1$, $d_v = 2$ and $y_2, y_3 \in \mathcal{R} \cap \mathcal{G}$.

**Proof.** A result of Jones [6] implies that if $p_0 > 0, p_1 \leq 0$ and if (1.1) has oscillatory solutions, then any nontrivial nonoscillatory solution $y(t)$ satisfies $\lim_{t \to \infty} y(t) = 0$. Therefore, if $\lambda_1 > 0$ is a positive multiplier, then since the corresponding solution $y_1(t) = y_{\lambda_1}(t) = \varphi(t)e^{\rho t} \in \mathcal{R}$, it follows that $\rho_1 = \ldots$
In $\lambda_1/\omega < 0$, $0 < \lambda_1 < 1$, so that $\lambda_2, \lambda_3$ are both negative or complex. In any case, the corresponding solutions $\gamma_2, \gamma_3 \in \mathcal{O}$. If Condition A holds, then $\lambda_2, \lambda_3$ must be complex so that $|\lambda_2| = |\lambda_3| > 1$. This completes the proof.

3. Concluding remarks. In this section we wish to briefly discuss the applicability of the results of §2.

If $p_0 > 0$ and $3p_1 > p_2^2$, then $\sigma_p(t, \rho) > 0$ for all $\rho$ and hence $\sigma(t, \rho) > 0$ for all $\rho > 0$ so that $\alpha_1 < 0$. If $p_0 < 0$ and $3p_1 > p_2^2$, then $\sigma(t, \rho) < 0$ for all $\rho < 0$ so that $\alpha_1 > 0$. Note also that $\sigma(t, \rho) > p_2^2 + p_1 + p_0 > 0$ for all $\rho > 0$ provided $p_0 > 0, p_2 > 0$ and $4p_0p_2 > p_1^2$. Thus, $\beta_1 < 0$ in this case also. Similarly, $\sigma(t, \rho) < 0$ for all $\rho < 0$ if $p_0 < 0, p_2 < 0$ and $4p_0p_2 > p_1^2$ so that $\alpha_1 > 0$ in this case. Likewise, $0 < \alpha_1 < \beta_1 < 1$ provided $\sigma(t, 0) = p_0 < 0 < \sigma(t, 1) = 1 + p_2 + p_1 + p_0$ and $\sigma_0(t, \rho) > 0$ for $\rho < 0$ and $\rho > 1$. Analogously, $-1 < \alpha_1 < \beta_1 < 0$ if $\sigma(t, 0) = p_0 > 0 > \sigma(t, -1) = -1 + p_2 - p_1 + p_0$ and $\sigma_0(t, \rho) < 0$ for $\rho < -1$ and $\rho > 0$. Finally, we note that $\beta_2 < -1 < q_2 < 0$ if $\sigma(-1, t) = -1 + p_2 - p_1 + p_0 > 0$ and $\sigma(1, t) = 1 + p_2 + p_1 + p_0 < 0$. This holds if $1 + p_1 < p_2 + p_0 < -(1 + p_1)$.

Sufficient conditions for Condition A to hold and for disconjugacy may be found in [12]. Other disconjugacy tests may be found in [1], [4], [7], [9], [11]. These tests combined with the remarks of the preceding paragraph allows one to give examples illustrating the results of §2. We leave this to the interested reader.

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