SHORTER NOTES

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A SIMPLE PROOF ABOUT POSITIVE HARMONIC FUNCTIONS ON $\mathbb{R}^n$

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Abstract. A simple proof is given that positive harmonic functions on $\mathbb{R}^n$ are constant.

The following is a simple elementary proof that positive harmonic functions on $\mathbb{R}^n$ are constant.

Let $f(x)$ be a nonconstant positive harmonic function. Choose $x$ in the unit sphere such that $f(x) - f(0) = c > 0$. Let $B(x, n)$ and $B(0, n)$ be balls of radius $n$ centered at $x$ and $0$ respectively. Let $C_n = B(x, n) \setminus B(0, n)$ and $D_n = B(x, n) \cap B(0, n)$. Notice that $C_m \cap C_n = \emptyset$ if $m \neq n$, also that $|B(x, n)| = |B(0, n)|$ where $| \cdot |$ is the volume. By the mean value theorem,

$$c = f(x) - f(0) = \frac{\int_{B(x,n)} f}{|B(x,n)|} - \frac{\int_{B(0,n)} f}{|B(0,n)|}$$

$$\leq \frac{\int_{C_n} f + \int_{D_n} f}{|B(x,n)|} - \frac{\int_{D_n} f}{|B(0,n)|} = \frac{\int_{C_n} f}{|B(x,n)|}.$$ 

Hence $c|B(x, n)| < \int_{C_n} f$.

We will show by the mean value theorem again and the disjointness of $C_n$'s that $f(x) = \infty$, thus arriving at a contradiction as follows:

$$f(x) = \frac{\int_{B(x,n)} f}{|B(x,n)|} > \frac{\sum_{i=1}^{n} |C_i f|}{|B(x,n)|} > \frac{c \sum_{i=1}^{n} |B(x,i)|}{|B(x,n)|} \to \infty, \quad \text{as } n \to \infty$$

by an easy estimate.

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