ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A FOURTH ORDER NONLINEAR DIFFERENTIAL EQUATION

W. E. TAYLOR, JR.

ABSTRACT. In this paper, the asymptotic properties of solutions of a certain fourth order differential equation are considered. Sufficient conditions for oscillation are also given.

Introduction. This paper is concerned with the solutions of the differential equation

\[(y'' + p(x)y)' + p(x)y' + f(y) = 0\]

where \(p(x)\) is a continuously differentiable function defined on \([0, \infty)\) satisfying \(\int_0^\infty p(x)\,dx = \infty\). The function \(f: (-\infty, \infty) \to (-\infty, \infty)\) is assumed to be continuous and satisfy the condition \(f(y)/y > m > 0\) for \(y \neq 0\). Under these assumptions, continuable nontrivial solutions of (1) with a multiple zero are oscillatory. (See Theorem 4.)

The motivation for this study comes from a recent article by D. L. Lovelady [2]. In [2], Lovelady considers a special class of nonlinear fourth order equations and derives some oscillation criteria. We also refer to the works of J. W. Heidel [1] and P. Waltman [3] on nonlinear third order differential equations. Unlike the results in [1] and [3], we do not require \(p(x)\) to remain one-signed in most of our results.

A solution \(y(x)\) of (1) is said to be continuable if it exists on some ray \([a, \infty), \ a > 0\). A nontrivial solution of (1) is oscillatory if it is continuable and has arbitrarily large zeros. By a nonoscillatory solution we mean a continuable solution which is not oscillatory. The term "solution" for the remainder of this work will mean a nontrivial continuable solution.

Main results. Our first result is essential to the results which follow.

Lemma 1. Let \(y(x)\) be a solution of (1). Then

\[F(y(x)) = y(x)[y'''(x) + p(x)y(x)] - y'(x)y''(x)\]

is nonincreasing on some ray \([a, \infty), \ a > 0\), in fact,
\[ (F[y(x)])' = -(y''(x))^2 - y(x)f(y(x)). \]

Since \( F[y(x)] \) is monotone it follows that \( F[y(x)] \) is one-signed on some ray \([c, \infty)\). Using this fact, we will call a solution \( u(x) \) of (1), type I, if \( F[u(x)] > 0 \) on some ray \([d, \infty)\). If \( F[u(c)] < 0 \) for some \( c > 0 \), then \( u(x) \) is said to be type II.

**Lemma 2.** Suppose \( f(x) \) is a twice continuously differentiable function on \([a, \infty)\) satisfying \( \int_a^\infty f''^2(x) \, dx < \infty \) and \( \int_a^\infty f^2(x) \, dx < \infty \). Then

\[
\left( \int_a^\infty f''^2(x) \, dx \right)^2 < \int_a^\infty |f|^2 \cdot \int_a^\infty |f''|^2.
\]

**Proof.** Expand \( f \) in a Fourier cosine series on \([a, a + T]\), use Parseval’s equality and the C-B-S inequality, then let \( T \to \infty \).

We now examine properties of type I solutions.

**Theorem 3.** Let \( y(x) \) be a type I solution. Then the following are true:

(i) \( \int_0^\infty y''^2(x) \, dx < \infty \) and \( \int_0^\infty y(x)f(y(x)) \, dx < \infty \),

(ii) \( \int_0^\infty y'^2(x) \, dx < \infty \),

(iii) \( \int_0^\infty y^2(x) \, dx < \infty \).

**Proof.** Since \( y(x) \) is type I, \( F[y(x)] > 0 \) on \([a, \infty)\) for some \( a > 0 \). By differentiating \( F[y(x)] \) and integrating from \( a \) to \( x \) we obtain

\[ 0 \leq F[y(x)] = F[y(a)] - \int_a^x y''^2(t) \, dt - \int_a^x y(t)f(y(t)) \, dt. \]

This proves (i).

To prove (ii), note that \( y(x)f(y(x)) \geq my^2(x) \), and apply (i).

Finally, the proof of (iii) follows immediately from (i), (ii) and Lemma 2.

We now consider the type II solutions.

**Theorem 4.** Let \( y(x) \) be a type II solution. Then \( y(x) \) is oscillatory.

**Proof.** Suppose \( y(x) \) is eventually positive, then there exists \( x = c \) such that \( y(x) > 0 \) on \([c, \infty)\) and \( F[y(c)] < 0 \).

Consider the function

\[ J(x) = \frac{y''(x)}{y(x)} + \int_c^x p(t) \, dt. \]

By differentiating \( J(x) \) we find

\[ J'(x) = \frac{y''(x)}{y(x)} + \int_c^x p(t) \, dt. \]

for \( x > c \). So \( J(x) \) is decreasing on \([c, \infty)\). Since \( \int_c^\infty p(x) \, dx = \infty \), it follows that \( y''(x) < 0 \) for large \( x \). Since \( y(x) > 0 \) we must have \( y'(x) > 0 \) on some ray \([d, \infty)\), \( d > c \). The fact that \( \int_c^\infty p(t) \, dt \to \infty \) as \( x \to \infty \) and \( J(x) \) is decreasing implies \( y''(x)/y(x) \to -\infty \) as \( x \to \infty \). But this implies \( y''(x) \) is
bounded away from zero for large $x$, implying $y(x) \to -\infty$ as $x \to \infty$. This contradiction proves the theorem.

**Corollary.** Any nontrivial solution of (1) with a multiple zero is oscillatory.

**Lemma 5.** Suppose $p(x) > 0$ and let $y(x)$ be a type II solution. Then

$$N[y(x)] = y(x)y''(x) - y'^2(x) \to -\infty$$

as $x \to \infty$.

**Proof.** Note that $N'[y(x)] = F[y(x)] - p(x)y^2(x) < F[y(x)]$. Since $y(x)$ is a type II solution, $F[y(x)]$ is negative and bounded away from zero on some ray $[a, \infty)$ and the result follows.

As our final theorem we list some properties of type II solutions.

**Theorem 6.** Let $y(x)$ be a type II solution and assume $p(x) > 0$. Then
1. $y'(x)$ is unbounded, and
2. $\int_a^\infty y'^2(x)\,dx = \infty$.

**Proof.** From Lemma 5, $N[y(x)] \to -\infty$ as $x \to \infty$. Since $y(x)$ is oscillatory (Theorem 4) (i) follows immediately by evaluating $N[y(x)]$ along the zeros of $y(x)$.

To prove (ii), integrate $N[y(t)]$ from $c$ to $x$ where $y'(c) = 0$. Doing so, we obtain

$$\int_c^x N[y(t)]\,dt = y(x)y'(x) - 2 \int_c^x y'^2(t)\,dt.$$

But $\int_c^x N[y(t)]\,dt \to -\infty$ as $x \to \infty$ and, since $y(x)$ is oscillatory, (ii) follows.

**References**