ON GENERIC ASYMPTOTIC STABILITY OF DIFFERENTIAL EQUATIONS IN BANACH SPACE

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Abstract. The asymptotic stability of the zero solution of the differential equation \((*)\) \(x' = Ax + f(x)\) is studied, when the perturbation \(f\) is in a given complete metric space \(\mathcal{M}\). It is known that the zero solution of \((*)\) is asymptotically stable whenever \(f\) is in a certain proper subset \(\mathcal{H} \subset \mathcal{M}\). It is shown that, while \(\mathcal{H}\) is of Baire first category in \(\mathcal{M}\), on the contrary the set \(\mathcal{H}_0\) of all those \(f\) for which the zero solution of \((*)\) is asymptotically stable is a proper residual subset of \(\mathcal{M}\).

1. Introduction. Denote by \(E\) a Banach space with norm \(\| \cdot \|\). We shall consider the linear stationary equation

\[(L)\quad x' = Ax \quad \left( ' = \frac{d}{dt} \right)\]

where \(A : E \to E\) is a linear bounded operator whose spectrum lies in the interior of the left halfplane, so that

\[(P)\quad \|e^{At}\| \leq ke^{-\alpha t} \quad \text{for all } t \in \mathbb{R}^+ \quad (\alpha > 0, 1 \leq k < +\infty).\]

We associate with \((L)\) the perturbed equation

\[x' = Ax + f(x)\]

in which the perturbing term \(f\) is supposed belonging to some metric space to be specified.

Denote by \(B_r\) the open ball in \(E\) with center the origin and radius \(r > 0\). Define

\[\mathcal{M} = \{f : B_r \to E | f(0) = 0, \|f(x) - f(y)\| \leq \gamma \|x - y\|, \gamma k / \alpha < 1\},\]

\[\mathcal{N} = \{f \in \mathcal{M} | \gamma k / \alpha < 1\}.\]

\(\mathcal{M}\), endowed with the distance \(\|f - g\| = \sup\{\|f(x) - g(x)\| : x \in B_r\}\), becomes a complete metric space. Observe that \(\mathcal{M}\) is convex.

Under the hypotheses made over \(A\) the zero solution of \((L)\) is asymptotically stable. This property remains valid for \((P)\) provided that the perturbing term \(f\) is sufficiently small, in particular if \(f \in \mathcal{N}\) [5, p. 504] but it is, in general, lost if it is assumed \(f \in \mathcal{H}\). In this case all one can say is that the origin is merely stable for \((P)\).
If $E$ is a Hilbert space the set $\mathcal{R}$ turns out to be small (in the sense of the Baire category) with respect to $\mathcal{M}$, that is $\mathcal{R}$ is of first category in $\mathcal{M}$. Denote by $\mathcal{M}_0$ the set of all $f \in \mathcal{M}$ such that the origin is asymptotically stable for (P). Clearly $\mathcal{M}_0 \subseteq \mathcal{M}$. We might then expect that $\mathcal{M}_0$ is only a bit larger than $\mathcal{M}$ itself, namely that $\mathcal{M}_0$ is of first category in $\mathcal{M}$. In the present note it will be shown just the opposite. The set $\mathcal{M}_0$ is in fact residual in $\mathcal{M}$ (see Theorem 2). This result can be read in another way: with respect to the class $\mathcal{M}$ of admissible perturbations, the hypothesis $f \in \mathcal{M}$, though sufficient to guarantee the asymptotic stability of the zero solution of (P), is far away from being a necessary condition as well. To prove the aforementioned theorem we shall use, in the appropriate form, some ideas which go back to Orlicz [4] and which have been further developed in a number of recent papers (see [7] and [1], [2]).

2. Preliminary lemmas. In the sequel to emphasize the presence of the perturbation $f$ in the differential equation (P) we shall denote this equation, for a given perturbation $f$, by $[A, f]$. We agree that saying $[A, f]$ is stable (asymptotically stable) means that the zero solution of $[A, f]$ is stable (asymptotically stable).

**Remark.** If $f \in \mathcal{M}$ and $x_0 \in B_r$, $[A, f]$ has a unique solution $x^f(t; x_0)$ which satisfies $x^f(0; x_0) = x_0$ [3]. It follows from the proof of the next lemma that, for every $f \in \mathcal{M}$,

$$x_0 \in B_{r/k} \Rightarrow \|x^f(t; x_0)\| < r \quad \text{for all } t \in \mathbb{R}^+,$$

that is each solution of $[A, f]$ which starts in $B_{r/k}$ is defined for all $t \in \mathbb{R}^+$ and does not leave $B_r$.

**Lemma 1.** For every $f \in \mathcal{M}$, $[A, f]$ is stable.

**Proof.** Let $f \in \mathcal{M}$ and $e > 0$ ($e < r$). We wish to show that there is $\delta > 0$ ($\delta < r$) such that

$$\|x_0\| < \delta \Rightarrow \|x^f(t; x_0)\| < e \quad \text{for all } t \in \mathbb{R}^+.$$

By Lagrange formula

$$x^f(t; x_0) = e^{At}x_0 + \int_0^t e^{A(t-s)}f(x^f(s; x_0)) \, ds$$

and

$$\|x^f(t; x_0)\| \leq \|e^{At}\| \|x_0\| + \int_0^t \|e^{A(t-s)}\| \|f(x^f(s; x_0))\| \, ds$$

$$< k e^{-\alpha t} \|x_0\| + \gamma k \int_0^t e^{-\alpha(t-s)} \|x^f(s; x_0)\| \, ds.$$

Multiplying both sides of the last inequality by $e^{\alpha t}$, hence using Gronwall's inequality we get

$$\|x^f(t; x_0)\| e^{\alpha t} < k \|x_0\| e^{\gamma t}.$$
that is
\[ \|x'(t; x_0)\| < k\|x_0\|e^{(\gamma k - a)t}. \]
Since \( \gamma k - a < 0 \) the claim follows at once if we let \( \delta < \varepsilon/k \).

**Example.** Associate with the stationary equation \( x' = -ax \) \((a > 0)\) the perturbed equation given by
\[ x' = -ax + bx \]
where \( b \) is a real number. Since in this example \( k = 1 \) and \( a = a \), the perturbation term \( bx \) is certainly in \( \mathbb{R} \) provided that \( |b| < a \). Notice that, if \( b = -a \), the origin is asymptotically stable for (2) while, if \( b = a \), the origin is only stable (but not asymptotically stable).

**Lemma 2.** The set \( \mathbb{R} \) is dense in \( \mathbb{R} \).

**Proof.** Let \( f \in \mathbb{R} \) and denote by \( \gamma \) the corresponding Lipschitz constant. Let \( \varepsilon > 0 \). Set \( g_\lambda = \lambda f \), where \( 0 < \lambda < 1 \), and observe that \( g_\lambda \) has Lipschitz constant equal to \( \lambda \gamma \), thus \( g_\lambda \in \mathbb{R} \). Then
\[ \| g_\lambda - f \| = \sup_{x \in B_r} \| \lambda f(x) - f(x) \| \leq (1 - \lambda)ar, \]
and \( \| g_\lambda - f \| < \varepsilon \) if \( \lambda > 1 - \varepsilon/ar \).
Denote by \( S(f, \varepsilon) \) the open ball in \( \mathbb{R} \) with center \( f \) and radius \( \varepsilon > 0 \).

**Lemma 3.** Let \( g \in \mathbb{R} \) and \( \varepsilon > 0 \) (\( \varepsilon < r \)). Then there exists \( \delta = \delta_g(\varepsilon) > 0 \) such that, for every \( x_0 \in B_{r/k} \),
\[ f \in S(g, \delta) \Rightarrow \|x'(t; x_0) - x^g(t; x_0)\| < \varepsilon \text{ for all } t \in \mathbb{R}^+, \]
where \( x^f(\cdot; x_0) \) and \( x^g(\cdot; x_0) \) are solutions of \([A, f]\) and \([A, g]\) respectively, with initial point \( x_0 \).

**Proof.** After the Remark any solution of \([A, f]\) with \( f \in \mathbb{R} \), which starts in \( B_{r/k} \) remains in \( B_r \) for every \( t \in \mathbb{R}^+ \). From (1) and the analogous equation for \( x^g(\cdot; x_0) \) we obtain
\[ \|x'(t; x_0) - x^g(t; x_0)\| \leq \int_0^t \|e^{A(t-s)}\| \|f(s; x_0) - g(s; x_0)\| ds \]
\[ + \int_0^t \|e^{A(t-s)}\| \|g(s; x_0) - g(s; x_0)\| ds \]
\[ \leq \delta k \int_0^t e^{-\alpha(t-s)} ds + \gamma k \int_0^t e^{-\alpha(t-s)} \|x^f(s; x_0) - x^g(s; x_0)\| ds, \]
where \( \gamma \) is the Lipschitz constant of \( g \). Then
\[ \|x'(t; x_0) - x^g(t; x_0)\| e^{\alpha t} \leq \frac{\delta k}{\alpha} e^{\alpha t} + \gamma k \int_0^t \|x^f(s; x_0) - x^g(s; x_0)\| e^{\alpha t} ds \]
and, by Gronwall's inequality,
\[ \|x'(t; x_0) - x^g(t; x_0)\| \leq \frac{\delta k}{\alpha} e^{\alpha t} + \gamma k \int_0^t e^{\alpha t} e^{\gamma k(t-s)} ds \]
which furnishes
\[ \|x^f(t; x_0) - x^g(t; x_0)\| \leq \delta \frac{k}{\alpha - \gamma k}, \quad \text{for all } t \in \mathbb{R}^+. \]
Since \( \alpha - \gamma k > 0 \) to complete the proof it suffices choosing \( \delta \) such that
\[ 0 < \delta < \frac{(\alpha - \gamma k)}{k}. \]

3. Main results.

**Theorem 1.** Let \( E \) be a Hilbert space. Then the set \( \mathcal{N} \) is of first category in \( \mathcal{N} \).

**Proof.** Let \( \{\gamma_i\}, 0 < \gamma_i < k/\alpha \), be an increasing sequence of reals such that \( \gamma_i \to k/\alpha \) as \( i \to \infty \). Define
\[ \mathcal{N}_i = \{ f \in \mathcal{N} : \|f(x) - f(y)\| \leq \gamma_i \|x - y\| \}. \]
It is clear that \( \mathcal{N}_i \) is closed and \( \mathcal{N} = \bigcup_{i=1}^{\infty} \mathcal{N}_i \). We claim that
\[ \text{int } \mathcal{N}_i = \emptyset, \quad i = 1, 2, \ldots. \]
Otherwise, for some \( i \), there exist \( f \in \mathcal{N}_i \) and \( \varepsilon > 0 \) such that \( S(f, \varepsilon) \subset \mathcal{N}_i \). We shall prove that in this sphere there exists at least one function \( g \in \mathcal{N} \) which is not in \( \mathcal{N}_i \).

Choose \( \eta \) such that \( 0 < \eta < \min\{r, \varepsilon k/2\alpha\} \) and set \( \sigma = (\alpha - \gamma_i k)\eta/2\alpha \).
Define
\[ \tilde{g}(x) = \begin{cases} \alpha x/k, & x \in B_\sigma, \\ f(x), & x \in B_r \setminus B_\sigma. \end{cases} \]
We shall show that \( \tilde{g} \) satisfies the Lipschitz condition
\[ \|\tilde{g}(x) - \tilde{g}(y)\| \leq \frac{\alpha}{k} \|x - y\| \quad \text{on } B_\sigma \cup (B_r \setminus B_\sigma). \]
To see this it is sufficient to verify the above inequality for \( x \in B_\sigma \) and \( y \in B_r \setminus B_\sigma \). For such choice of \( x \) and \( y \) we have
\[ \|\tilde{g}(y) - \tilde{g}(x)\| = \|f(y) - \alpha x/k\| \leq \frac{\alpha}{k} \|x\| + \|f(y) - f(0)\| \leq \frac{\alpha}{k} \|x\| + \gamma_i \|y\| \leq \frac{\alpha}{k} \|x\| + \gamma_i \|y\| \leq \alpha(\|y\| - \sigma)/k \leq \alpha \|y - x\|/k. \]

By virtue of a theorem of Valentine [6] there exists an extension \( g \) of \( \tilde{g} \) which is defined all over \( B_\sigma \), and is there Lipschitzian with the same constant \( \alpha/k \). Obviously \( g \in S(f, \varepsilon) \) and \( g \notin \mathcal{N}_i \), a contradiction. This completes the proof.

Define
\[ \mathcal{N}_0 = \{ f \in \mathcal{N} : [A, f] \text{ is asymptotically stable} \}. \]

**Theorem 2.** The set \( \mathcal{N}_0 \) is residual in \( \mathcal{N} \).

**Proof.** Define \( V : \mathcal{N} \to \mathbb{R}^+ \) by
\[ V(f) = \sup_{x_0 \in B_r/k} \lim_{t \to \infty} \|x^f(t; x_0)\|, \]
where \( x^f(t; x_0) \) is the solution of \([A, f]\) through \( x_0 \). The functional \( V \) has the properties: (a) \( V(f) = 0 \) for each \( f \in \mathcal{R} \); (b) If \( f \in \mathcal{R} \) and \( V(f) = 0 \), then \([A, f]\) is asymptotically stable; (c) Let \( g \in \mathcal{R} \). Then for every \( f \in S(g, \delta_g(1/n)) \) we have \( V(f) < 1/n \).

Indeed, (a) and (b) are obvious while (c) follows immediately from the inequality
\[
\|x^f(t; x_0)\| \leq \|x^f(t; x_0) - x^g(t; x_0)\| + \|x^g(t; x_0)\|
\]
by virtue of Lemma 3 and the fact that \([A, g]\) is asymptotically stable.

Now define
\[
\mathcal{M}_1 = \bigcap_{n=1}^{\infty} \bigcup_{g \in \mathcal{R}} S\left(g, \delta_g\left(\frac{1}{n}\right)\right).
\]
Observe that, for every \( f \in \mathcal{M}_1 \), \( V(f) = 0 \) thus \([A, f]\) is asymptotically stable. Clearly \( \mathcal{M}_1 \) is dense in \( \mathcal{M} \) since \( \mathcal{M}_1 \supset \mathcal{R} \). On the other hand \( \mathcal{M}_1 \) is a \( G_\delta \) subset of \( \mathcal{M} \) and, \( \mathcal{M} \) being a complete metric space, it follows that \( \mathcal{M}_1 \) is a residual set in \( \mathcal{M} \). Since \( \mathcal{M}_0 \supset \mathcal{M}_1 \), the proof is complete.

4. Concluding remarks. The results of the preceding section suggest the following definition of stability.

Consider the perturbed equation \([A, f]\) and suppose that the perturbing term \( f \) is in some complete metric space \( \mathcal{T} \).

**Definition.** The zero solution of \([A, f], f \in \mathcal{T}\), is said to be generically stable (generically asymptotically stable) on \( \mathcal{T} \) if the set of all \( f \in \mathcal{T} \) for which \([A, f]\) admits the zero solution and this solution is stable (asymptotically stable) is residual in \( \mathcal{T} \).

Then Theorem 2 can be rephrased as follows. The zero solution of \([A, f], f \in \mathcal{M}\) is generically asymptotically stable on \( \mathcal{M} \).

**References**


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