A REMARKABLE CLASS OF CONTINUED FRACTIONS

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Abstract. For any irrational number \( \alpha \) and integer \( a > 1 \), the continued fraction of \( (a-1)^2 / a^a \) is computed explicitly in terms of the continued fraction of \( \alpha \).

1. Introduction. In [2], the second author proved the remarkable result that
\[
2^{1/2} = \frac{1}{2 + 1/a}
\]
where \( \alpha = (1 + \sqrt{5})/2 \) has as its continued fraction
\[ [0, 1, t_2, t_3, \ldots] \]
where \( t_n = 2^{f_{n-2}} (n > 2) \) and \( f_n \) is the \( n \)-th Fibonacci number.
In this paper we generalize this result.

Let \( \alpha > 0 \) be any real number and let \( a > 1 \) be an integer.\(^2\) Define
\[
S_a(\alpha) = (a-1) \sum_{r=1}^{\infty} \frac{1}{a^{[\alpha r]}}.
\]

Consider the continued fraction for \( \alpha^{-1} \),
\[
\alpha^{-1} = [a_0; a_1, a_2, \ldots].
\]
(We use the usual continued fraction notation, say for example in [3].) Let
\( p_n/q_n \) denote the convergents of \( \alpha^{-1} \). We adopt the convention throughout this paper that \( p_{-2} = 0, p_{-1} = 1, q_{-2} = 1, q_{-1} = 0 \). Then for all \( n > 0 \),
\[
p_n = a_n p_{n-1} + p_{n-2} \quad \text{and} \quad q_n = a_n q_{n-1} + q_{n-2}.
\]

Define another sequence of integers
\[
t_0 = a_0 a, \quad t_n = (a^{q_n} - a^{q_{n-2}})/(a^{q_{n-1}} - 1) \quad \text{for} \ n > 1.
\]
(\( t_n \) is an integer since \( q_{n-1}q_n - q_{n-2} \) is an integer since \( q_{n-1}|a^n - a^{q_{n-2}} \)).

Theorem. If \( \alpha \) is irrational and \( \alpha > 0 \), then
\[
S_a(\alpha) = [t_0; t_1, t_2, \ldots].
\]

We will also give explicit formulas for convergents \( P_n/Q_n \) of \( S_a(\alpha) \). For this we need to define for real \( x \),
\[
[x] = \begin{cases} 
\left\lfloor x \right\rfloor, & \text{if} \ x \ \text{is not an integer,} \\
\lfloor x \rfloor - 1, & \text{if} \ x \ \text{is an integer,}
\end{cases}
\]

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\(^1\) The results in this paper were obtained independently by the two authors. The second author would like to thank M. Mendes France for pointing out that the "2" in his first paper could be replaced by a variable "a".
\(^2\) Any \( a \) with \( |a| > 1 \) yields the same result.
the largest integer < \( x \). Further define

\[
S'_a(x) = (a - 1) \sum_{r=1}^{\infty} \frac{1}{a^{\lfloor r \alpha \rfloor}}.
\]

(Of course, for irrational \( \alpha \), \( S_a(\alpha) = S'_a(\alpha) \)).

**Proposition.** If \( \alpha \) is irrational, \( p_n/q_n \) are the convergents of \( \alpha^{-1} \) and \( P_n/Q_n \) are the convergents of \( S_a(\alpha) \), then for \( n > 0 \),

\[
\frac{P_n}{Q_n} = \begin{cases} 
S'_a(q_n/p_n), & \text{if } n \text{ even}, \\
S_a(q_n/p_n), & \text{if } n \text{ odd}
\end{cases}
\]

(if \( p_0 = 0 \) interpret \( S'_a(q_0/p_0) = 0 \)).

Also

\[
P_n = \begin{cases} 
\sum_{t=1}^{p_n} a_{n+1-t} q_n/p_n, & \text{if } n \text{ even}, \\
\sum_{t=1}^{p_n} a_{n+1-t} q_n/p_n, & \text{if } n \text{ odd},
\end{cases}
\]

and

\[
Q_n = (a_{n+1} - 1)/(a - 1).
\]

In the last section we observe that \( S_a(\alpha) \) is transcendental.

2. **Lemmas.** We assume throughout that \( \alpha \) is irrational.

**Lemma 1.** Assume \( 1 < r < p_{n+1} \) and \( n > 0 \). Then

\[
\lfloor r \alpha \rfloor = \begin{cases} 
[rq_n/p_n], & \text{if } n \text{ even}, \\
[rq_n/p_n], & \text{if } n \text{ odd}
\end{cases}
\]

(for \( n = 0 \) assume \( p_0 \neq 0 \)).

**Proof.** The \( n \)-th convergent to \( \alpha^{-1} \) is \( p_n/q_n \) and thus \( q_n/p_n \) is the \( n+1 \) or \( n-1 \) (depending on whether \( a_0 = 0 \) or \( a_0 \neq 0 \), respectively) convergent of \( \alpha \). Thus

\[
\alpha = q_n/p_n + \theta_n/p_{n+1},
\]

where \( 0 < \theta_n < 1 \) if \( n \) is odd and \( -1 < \theta_n < 0 \) if \( n \) is even. Hence, \( 1 < r < p_{n+1} \) implies

\[
r\alpha = rq_n/p_n + \theta'_n/p_n,
\]

where \( 0 < \theta'_n < 1 \) if \( n \) is odd and \( -1 < \theta'_n < 0 \) if \( n \) is even. If \( p_n \nmid r \), then there is no integer between \( rq_n/p_n \) and \( r\alpha \) and

\[
\lfloor r \alpha \rfloor = \lfloor rq_n/p_n \rfloor = \lfloor rq_n/p_n \rfloor'.
\]

If \( p_n \mid r \), then

\[
\lfloor r \alpha \rfloor = rq_n/p_n + \left\lfloor \theta'_n/p_n \right\rfloor,
\]

which immediately gives the desired result.
Lemma 2. Assume \( 1 < r < p_n \) and \( n > 0 \). Then
\[
[r_{q_n}/p_n] = [r_{q_{n-1}}/p_{n-1}]', \quad n \text{ odd},
\]
\[
[r_{q_n}/p_n]' = [r_{q_{n-1}}/p_{n-1}], \quad n \text{ even}.
\]

Proof. Immediate from Lemma 1.

Lemma 3.
\[
\lim_{n \to \infty} S_a \left( \frac{q_n}{p_n} \right) = \lim_{n \to \infty} S_a' \left( \frac{q_n}{p_n} \right) = S_a(\alpha).
\]

Proof. If \( n \) is even, we have by Lemma 1,
\[
\left| S_a' \left( \frac{q_n}{p_n} \right) - S_a(\alpha) \right| = (a - 1) \left| \sum_{r=p_{n+1}}^{\infty} \left( a^{-[r_{q_n}/p_n]} - a^{-[r_{\alpha}]} \right) \right|
\]
and it is trivial to see that this tends to zero. The situation is similar if \( n \) is odd.

Lemma 4.

(i) \[
S_a \left( \frac{q_n}{p_n} \right) = \frac{a - 1}{a^{q_n} - 1} \sum_{t=1}^{p_n} a^{-[r_{q_n}/p_n]}.
\]

(ii) \[
S_a' \left( \frac{q_n}{p_n} \right) = \frac{a - 1}{a^{q_n} - 1} \sum_{t=1}^{p_n} a^{-[r_{q_n}/p_n]}.
\]

Proof. In definition (3) of \( S_a'(q_n/p_n) \), set \( r = kp_n + t, 1 < t < p_n, 0 < k < \infty \). Thus
\[
S_a' \left( \frac{q_n}{p_n} \right) = (a - 1) \sum_{k=0}^{\infty} \sum_{t=1}^{p_n} a^{-[r_{q_n}/p_n]} \cdot \sum_{k=0}^{\infty} a^{-kq_n}
\]
\[
= (a - 1) \sum_{t=1}^{p_n} a^{-[r_{q_n}/p_n]} \sum_{k=0}^{\infty} a^{-kq_n}
\]
\[
= (a - 1) \left( \sum_{t=1}^{p_n} a^{-[r_{q_n}/p_n]} \right) \left( 1 - \frac{1}{a^{q_n}} \right)^{-1}
\]
which is the desired result. \( S_a(q_n/p_n) \) is similar.

3. Proof of the theorem and proposition. From Lemmas 3 and 4 we see that it suffices to show that the quantities on the right-hand sides of (4) and (5) satisfy the appropriate recursion relations. Let \( P_n' \) denote the right side of (4) and \( Q_n' \) the right side of (5) \( (n > 0) \).

Set \( Q_{-2}' = 1, Q_{-1}' = 0 \). We need that for all \( n > 0, \)
\[
Q_n' = t_n Q_{n-1}' + Q_{n-2}'.
\]
where \( t_n \) is given in (2). For \( n = 0 \) this is clear. For \( n = 1, \)

\[3 \text{ This theorem was already proved by Loxton and Van der Poorten (see [4], [5]). Their proof resembles, in much less explicit form, our proof for the continued fraction identity.} \]
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\[ t_1 Q'_0 + Q'_{-1} = t_1 = \frac{(a^{q_1} - a^{q_0})}{(a^{q_0} - 1)} = Q'_1. \]

For \( n > 2 \), we have by induction

\[ t_n Q'_{n-1} + Q'_{n-2} = \frac{a^{q_n} - a^{q_{n-2}}}{a^{q_{n-1}} - 1} + \frac{a^{q_{n-2}} - 1}{a - 1} = \frac{(a^{q_n} - 1)(a - 1)}{a - 1} = Q'_n \]
as desired.

Set \( P'_{-2} = 0, P'_{-1} = 1 \). We need that for all \( n > 0 \),

\[ P'_n = t_n P'_{n-1} + P'_{n-2}. \]

If \( n = 0 \), then \( t_0 P'_{-1} + P'_{-2} = t_0 = a_0 a_1 \), whereas

\[ P'_0 = \sum_{i=1}^{P_0} a^{q_i - \lfloor q_i/p_0 \rfloor} = \sum_{i=1}^{a_0} a^{1-\lfloor 1/a_0 \rfloor} = a_0 a. \]

(If \( a_0 = 0 \), this is still valid.)

If \( n = 1 \),

\[ t_1 P'_0 + P'_{-1} = a_0 a \frac{a^{q_1} - a^{q_0}}{a^{q_0} - 1} + 1 = a_0 a \frac{a^{a_1} - 1}{a - 1} + 1. \]

Also

\[ P'_1 = \sum_{i=1}^{P_1} a^{q_i - \lfloor q_i/p_1 \rfloor} = \sum_{i=1}^{P_1 - 1} a^{q_i - \lfloor q_i/p_0 \rfloor} + 1 \]

\[ = \sum_{i=1}^{P_1 - 1} a^{a_{1-\lfloor 1/a_0 \rfloor} + 1,} \]

using Lemma 2. (If \( a_0 = 0 \) the result is clear.) It is easily seen that the numbers \( \lfloor t/a_0 \rfloor, 1 \leq t < p_1 - 1 = a_1 a_0 \), give \( a_0 \) repetitions of each of the numbers 0, 1, 2, \ldots, \( a_1 - 1 \). Hence

\[ P'_1 = a_0 \sum_{j=0}^{a_1 - 1} a^{a_{1-\lfloor 1/a_0 \rfloor} + 1} \]

which is the same as (7).

Now suppose \( n > 1 \). We verify (6) in the case that \( n \) is even; the case where \( n \) is odd is similar. From Lemma 2 and (4)

\[ P'_n = \sum_{i=1}^{P_n - 1} a^{q_i - \lfloor q_i/p_n \rfloor} + a. \]

Set \( i = \lfloor p_{n-1} + j, 1 \leq j < p_n - 1 \). Then

\[ 1 \leq i < p_n - 1 = a_n p_{n-1} + p_{n-2} - 1 \]

if and only if

\[ 0 < i < a_n - 1 \quad \text{and} \quad 1 \leq j < p_{n-1} \]
or

\[ i = a_n \quad \text{and} \quad 1 \leq j < p_{n-2} - 1. \]

Thus
\[ P_n' = \sum_{i,j} a^{q_n - iq_{n-1} - [q_{n-1}/p_{n-1}]} + a \]
\[ = \sum_{j=1}^{p_{n-1}} a^{q_n - [q_{n-1}/p_{n-1}]} \sum_{i=0}^{a_{n-1} - 1} a^{-iq_{n-1}} \]
\[ + \sum_{j=1}^{p_{n-2}-1} a^{q_n - a_{n-1}q_{n-1} - [q_{n-1}/p_{n-1}]} + a \]
\[ = \frac{a^{-a_{n-1}q_{n-1}} - 1}{a^{-q_{n-1}} - 1} \sum_{j=1}^{p_{n-1}} a^{q_n - [q_{n-1}/p_{n-1}]} + \sum_{j=1}^{p_{n-2}-1} a^{q_n - [q_{n-1}/p_{n-1}]} + a \]
\[ = \frac{a^{q_n} - a_{n-2}}{a^{q_{n-1}} - 1} \sum_{j=1}^{p_{n-1}} a^{q_n - [q_{n-1}/p_{n-1}]} \]
\[ + \sum_{j=1}^{p_{n-2}-1} a^{q_n - 2[p_{n-2}/p_{n-1}]} + a \quad \text{(by Lemma 2)} \]
\[ = t_n P_{n-1}' + P_{n-2}' \]

by induction. The proof is complete.

4. The transcendence of \( S_\alpha(a) \). From Roth’s theorem [1, p. 104] we have that if \( |q\beta - p| < 1/q^{1.4} \) has an infinite number of integer solutions \( q, p \), then \( \beta \) is transcendental. If \( A_n/B_n \) are the convergents of \( \beta \) then

\[ |B_n \beta - A_n| < 1/B_{n+1} \leq 1/B_n^{1.4} \]

provided \( \log B_{n+1}/\log B_n > 1.4 \). For \( B_n = Q_n \) above we have

\[ \log Q_{n+1}/\log Q_n \sim q_n/q_{n-1} \quad (n \to \infty). \]

It is easily checked that \( \max(q_{n+1}/q_n, q_n/q_{n-1}) > 1.5 \). Thus

Theorem. \( S_\alpha(a) \) is transcendental for all irrationals \( \alpha \).

References


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