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A COUNTABLE SELF-INJECTIVE RING IS QUASI-FROBENIUS[1]

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ABSTRACT. A countable dimensional self-injective algebra is Artinian. There is an application to self-injective twisted group algebras.

It has been known for some time that a countable self-injective ring is semilocal (see for example [8]). In this paper we show that such a ring is in fact quasi-Frobenius. Special cases of this result have been proved previously, for example if the ring is also regular [3] or if it is a group algebra [8]. My thanks to Ken Louden for his help in the preparation of this paper.

Unless stated otherwise, all rings are associative with a unity. If S is a subset of a ring R, we denote its left annihilator in R by $l_R(S)$.

THEOREM 1 (FAITH [1]). *A ring is quasi-Frobenius if it is right self-injective and satisfies the descending chain condition on left annihilators.*

PROPOSITION 2. *Let R be a subring of S. Suppose that S_S is injective, ${}_RS$ is flat and S_R is free. Then R_R is injective.*

PROOF. The proof is left to the reader.

THEOREM 3. *Every countable subring of a quasi-Frobenius ring is contained in a countable quasi-Frobenius subring. Conversely, if every countable subring of a ring is contained in a quasi-Frobenius subring, then the ring is quasi-Frobenius.*

PROOF. Suppose first that T is a quasi-Frobenius ring and A is a countable subring. We construct a sequence of subrings $A = R_0 \subset R_1 \subset R_2 \subset \cdots \subset T$ inductively as follows. Given R_k, consider all n-tuples $\{a_1, \ldots, a_n\}$ of elements of R_k as n ranges over the positive integers. If $a_n \in a_1T + \cdots + a_{n-1}T$ choose $x_1, x_2, \ldots, x_{n-1} \in T$ so that $a_n = a_1x_1 + \cdots + a_{n-1}x_{n-1}$. If $a_n \notin a_1T + \cdots + a_{n-1}T$, choose $x_n \in T$ so that $x_na_i = 0$, $i = 1, 2, \ldots, n - 1$, and $x_na_n \neq 0$. Now do the same for the left ideal generated by $a_1, a_2, \ldots, a_{n-1}$. Let R_{k+1} be the subring of T generated by R_k and all the x's obtained. Let $R = \bigcup_{i=1}^{\infty} R_i$. Clearly $A \subset R$ and R is a countable subring, so

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we need only show that R is quasi-Frobenius.

As T is right and left Artinian, R satisfies ACC and DCC on right and left annihilators. If I is a finitely generated right ideal of R, then, by construction, $r_R(l_R(I)) = I$. A dual result holds for left ideals. Thus R satisfies ACC on finitely generated right and left ideals and so is right and left Noetherian. As R satisfies DCC on right and left annihilators it is right and left Artinian. Since R is right and left Artinian and satisfies the 'annihilator condition' [9, p. 276], R is quasi-Frobenius.

Now suppose that every countable subring of T is contained in a quasi-Frobenius subring. Then T is clearly right and left Artinian. In order to prove that T is right self-injective, we need only show that for all finitely generated right ideals I and J we have $r(l(I)) = I$ and $l(I \cap J) = l(I) + l(J)$, (see [9, p. 274]). However, if these conditions were not satisfied we could construct a countable subring A such that for any subring between A and T these would not be satisfied, and this contradicts the hypothesis that A is a subring of a quasi-Frobenius ring.

THEOREM 4. *Let R be a right self-injective ring and let $\{J_i\}_{i\in\Psi}$ be a descending chain of left annihilators, well-ordered by inclusion. Then the cardinality of Ψ is less than the cardinality of R.*

PROOF OF THE THEOREM. Suppose that the cardinality of Ψ is greater than or equal to the cardinality of R. We may suppose that Ψ is a set of ordinals. Let Φ be the set of ordinals strictly less than the cardinality of Ψ, thus $|\Phi| = |R|$, and we consider the descending chain of annihilators $\{J_i\}_{i\in\Phi}$. Suppose $R = \{a_j\}_{j\in\Phi}$. Suppose J_i annihilates the right ideal I_i on the left. Let $J = \bigcap_{i\in\Phi} J_i$ and let $I = \bigcup_{i\in\Phi} I_i$. Clearly J is the left annihilator of I. Consider the following proposition:

(P) For each ordinal $\alpha \in \Phi$ there is an element $b_\alpha \in I$ and an R-module map φ_α: $\Sigma_{\rho<\alpha} b_\rho R \to R$ such that

(1) If $\beta < \alpha$, then φ_α restricted to $\Sigma_{\rho<\beta} b_\rho R$ is φ_β,

(2) $\varphi_\alpha(b_\alpha) \neq a_\alpha b_\alpha$.

We prove (P) by transfinite induction. For $\alpha = 1$, choose $c_1 \in J_1$ so $c_1 - a_1 \not\in J$. Then choose $b_1 \in I$ so $(c_1 - a_1)b_1 \neq 0$. Let φ_1 be left multiplication by c_1.

Now suppose we have proved (P) for all ordinals less than δ. We have a right module homomorphism

$$\varphi'_\delta\colon \sum_{\rho<\delta} b_\rho R \to R,$$

simply given by the union of the φ_ρ, $\rho < \delta$. As R is right self-injective, φ'_δ is given by left multiplication, say by d_δ. Let x be an ordinal large enough so $\{b_j\}_{j<\delta} \subset I_x$. Choose $c_\delta \in J_x$ so $c_\delta + d_\delta - a_\delta \not\in J$, and then choose b_δ so that $(c_\delta + d_\delta - a_\delta)b_\delta \neq 0$. Define φ_δ to be left multiplication by $c_\delta + d_\delta$. Thus (P) is proved by transfinite induction.

Let φ: $\Sigma_{\rho\in\Phi} b_\rho R \to R$ be the right R-module map defined by the union of

the φ_ρ. Then for all $\alpha \in \Phi$, φ restricted to $\Sigma_{\rho<\alpha} b_\rho R$ is simply φ_α. Therefore $\varphi(b_\alpha) = \varphi_\alpha(b_\alpha) \neq a_\alpha b_\alpha$; hence, φ is not given by left multiplication, contradicting the hypothesis that R is right self-injective. This completes the proof of the theorem.

PROPOSITION 5. *Let A be an infinite set. Then there is a totally ordered (by inclusion) subset of the power set of cardinality* $2^{|A|}$.

The above proposition allows us to construct the following example. Let F be a countable field and let A be an infinite set of ordinals less than a given cardinality. Let $R_A = \Pi_{i \in A} F_i$ be the direct product of A copies of F. Then R is self-injective and $|R_A| = 2^{|A|}$. Also, R_A has a well-ordered descending chain of annihilators of cardinality $|A|$ and a totally ordered descending chain of annihilators of cardinality $2^{|A|}$. This example shows that 'well ordered' cannot be replaced by 'totally ordered' in the theorem.

THEOREM 6. *Let T be a right self-injective ring such that every countable subring is contained in a countable subring R, where T is free as a right R-module and flat as a left R-module. Then T is quasi-Frobenius.*

PROOF. By Proposition 2 and Theorem 3, it is enough to show that a countable right self-injective ring is quasi-Frobenius.

COROLLARY 7. *A countable dimensional self-injective algebra over a field is quasi-Frobenius.*

COROLLARY 8 (RENAULT). *A group algebra is self-injective only if the group is finite.*

PROOF. A self-injective group algebra is quasi-Frobenius, hence Artinian, so the group is finite.

COROLLARY 9. *A ring is quasi-Frobenius if and only if every countable subring is contained in a countable self-injective subring.*

PROOF. This is an easy consequence of Theorems 3 and 4.

If we look at rings without a unity, then most of the above theorems fail to hold. Let S denote the semigroup $\{e_1, e_2, \ldots : e_i e_j = e_j\}$. If F is any field, then the semigroup ring FS is left but not right self-injective and is neither right nor left Artinian.

Recall that a twisted group algebra F^tG is defined by a 2-cocycle t: $G \times G \to F - \{0\}$, where G is a group and F is a field, and where we define $\bar{g} \cdot \bar{h} = t(g, h)\overline{gh}$. Define the cocycle subfield of F to be the subfield generated by the image of t. Passman has constructed an example of an infinite group such that for certain fields the twisted group algebra is a field. In the same paper [6], Passman proved that if F is algebraically closed and uncountable and F^tG is Artinian, then G is finite. We use his idea in the following theorem.

THEOREM 10. *Suppose that F^tG is a self-injective twisted group algebra such*

that F is a proper extension of the algebraic closure of the cocycle subfield. Then G is finite.

PROOF. If G is not finite, then we may assume that it is countably infinite [7], hence F^tG is quasi-Frobenius. Let $\Delta(G)$ denote the set of elements in G with finitely many conjugates. Then $F^t\Delta(G)$ is self-injective, so $\Delta(G)$ is finite [7]. Now using an argument similar to Passman's [6, p. 648] we may assume that F^tG is Artinian and $\Delta(G) = \langle 1 \rangle$. Let K denote the cocycle subfield of F and let L denote the algebraic closure of K in F. Clearly

$$F^tG \cong F \otimes_L L^tG,$$

and as F is not algebraic over L, L^tG must be an algebraic L-algebraic [4]. By a Theorem of Passman, L^tG is a semiprime [5, p. 424], so L^tG is a semiprime Artinian algebraic algebra over an algebraically closed field. Therefore, G is finite.

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