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ON A HILL'S EQUATION WITH DOUBLE EIGENVALUES

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ABSTRACT. Let $\Delta(\lambda)$ be the discriminant of a Hill's equation with a π-periodic potential $q(x)$. It is shown that if $2 + \Delta(\lambda)$ has only double zeros then $q(x)$ is necessarily $\pi/2$-periodic.

We consider the Hill's equation

$$y'' + (\lambda - q(x))y = 0 \tag{1}$$

where $q(x + \pi) = q(x)$ and, without loss of generality, $\int_0^\pi q(x)\,dx = 0$; $q(x)$ will be assumed to be integrable on $(0, \pi)$. For much relevant background and other references see [2]. We shall denote by y_1, y_2, solutions of (1) satisfying

$$\begin{aligned} y_1(0) &= 1, \qquad y_1'(0) = 0, \\ y_2(0) &= 0, \qquad y_2'(0) = 1. \end{aligned} \tag{2}$$

The discriminant of Hill's equation is given by

$$\Delta(\lambda) = y_1(\pi) + y_2'(\pi)$$

and the zeros of $2 + \Delta(\lambda)$ are the eigenvalues of (1) subject to the boundary conditions

$$y(0) = y(\pi) = 0, \qquad y'(0) + y'(\pi) = 0. \tag{3}$$

THEOREM. *If all zeros of* $2 + \Delta(\lambda)$ *are double zeros, then* $q(x + \pi/2) = q(x)$ *a.e.*

REMARK. It is easy to show that if $q(x + \pi/2) = q(x)$, then necessarily all zeros of $2 + \Delta(\lambda)$ are double zeros.

This theorem was first proved by Borg [1] in a very long and difficult paper. The proof below has the advantage of shortness and is modeled on the proof of a finite dimensional analogue of this theorem for Hill's matrices [3].

Let $g(x, \varepsilon)$ be a Green's function satisfying

$$g'' + (\lambda - q(x))g = \delta(x - \varepsilon) \tag{4}$$

with the boundary conditions (3). A standard calculation now shows that

$$g(x) \equiv g(x, 0) = (y_2(x) + y_2(x - \pi))/(2 + \Delta(\lambda)). \tag{5}$$

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It is known [2] that $\Delta(\lambda)$ is an entire function of λ of order $\frac{1}{2}$ and type π. By standard asymptotic estimates

$$2 + \Delta(\lambda) = 2 + 2\cos\sqrt{\lambda}\ \pi + O\left(e^{|\sqrt{\lambda}\,\pi|}/\sqrt{\lambda}\right). \tag{6}$$

For $y_2(x)$ we have the estimate [2]

$$y_2(x) = \frac{\sin\sqrt{\lambda}\,x}{\sqrt{\lambda}} - \frac{\cos\sqrt{\lambda}\,x}{2\lambda}\int_0^x q(t)\,dt + O\left(\frac{e^{|\sqrt{\lambda}\,x|}}{\lambda^{3/2}}\right), \tag{7}$$

and as a function of λ, $y_2(x)$ is also entire and of order $\frac{1}{2}$.

Since all zeros of $2 + \Delta(\lambda)$ are double by hypothesis, we can let $2 + \Delta(\lambda) = f^2(\lambda)$ and rewrite $g(\pi/2)$ as

$$g\left(\frac{\pi}{2}\right) = \frac{\dfrac{y_2(\pi/2) + y_2(-\pi/2)}{f(\lambda)}}{f(\lambda)}. \tag{8}$$

The boundary value problem (1), (3) is selfadjoint, from which it follows that the Green's function can have only simple poles. Accordingly the numerator of (8) must be an entire function and, using the growth estimates (6) and (7), we find

$$(y_2(\pi/2) + y_2(-\pi/2))/f(\lambda) = O\left(1/\sqrt{\lambda}\right). \tag{9}$$

By Liouville's theorem, we can conclude that

$$y_2(\pi/2) + y_2(-\pi/2) = 0 \quad \text{for all } \lambda. \tag{10}$$

Using (7), (10) is equivalent to

$$-\frac{\cos\sqrt{\lambda}\,(\pi/2)}{\lambda}\int_0^{\pi/2} q(t)\,dt + O\left(\frac{1}{\lambda^{3/2}}\right) = 0$$

so that $\int_0^{\pi/2} q(t)\,dt = 0$.

The spectrum of (1), (3) remains unchanged if $q(x)$ is replaced by $q(x + \tau)$, where τ is an arbitrary translation. Then

$$\int_0^{\pi/2} q(t + \tau)\,dt = 0 \quad \text{for all } \tau. \tag{11}$$

(11) now shows that $q(x + \pi/2) = q(x)$ a.e.

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