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RELATIVE S-INVARIANTS

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ABSTRACT. Warfield has defined the concept of a T^*-module over a discrete valuation ring and has proved a classification theorem for these modules. In this paper, the invariant S defined by the author is extended. This allows a generalization of the classification theorem of Warfield.

1. **Preliminaries.** R will be a discrete valuation ring throughout, and p will represent a generator of its maximal ideal. The word "module" will mean R-module. Standard terminology of abelian groups will be used (see Fuchs [1], especially §79). The exception to this is that we will define the height of 0 to be ∞', and assume $\alpha < \infty < \infty'$ for all ordinals α. The indicator of an element x will be denoted $H(x)$. An indicator is called *proper* if it does not contain ∞'. If A is a submodule of M and α is an ordinal or ∞, the αth relative Ulm invariant will be denoted $f(\alpha, M, A)$.

In [5], (announced in [4]), Warfield defined a *T-module* as a module that can be defined in terms of generators and relations in such a way that the only relations are of the form $px = 0$ or $px = y$. A summand of a T-module is called a *T^*-module*. A module M has *torsion free rank one* if, for any two elements x and y of infinite order in M, there are nonzero elements r and s in R such that $rx = sy$. Warfield has shown that if M is a T-module, it is either a torsion module, or a direct sum of modules of torsion free rank one.

A subset X of a module M is a *decomposition basis* if $[X]$ is the free module on X, $M/[X]$ is torsion, and $h(\Sigma r_i x_i) = \min\{h(r_i x_i)\}$, for $r_i \in R$, $x_i \in X$. (Here $h(x)$ denotes the height of x.) It has been shown by Warfield that a T^*-module M has a decomposition basis X such that $[X]$ is nice in M.

In §2, we define the concept of stability, which is stronger than niceness. The invariant $S(e, M)$ defined in [2] is generalized. These concepts are used in §3 to generalize Warfield's theorem [5, Theorem 5.2].

2. **Stability and invariants.**

DEFINITION. Let M be a module and A be a submodule. A is called *stable* in M if:

(i) A is nice in M;

(ii) every coset $m + A$ of infinite order contains an element x such that $p^i x$

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is proper with respect to A for $i = 0, 1, 2, \ldots$. Such an element x is called *strongly proper* with respect to A.

We list without proof several elementary properties.

(A) A nice submodule A is stable in M exactly if $\mu(M/A) = (\mu M + A)/A$ for all proper indicators μ.

(B) If x is strongly proper with respect to A, then $H(x + a) = \min\{H(x), H(a)\}$ for all $a \in A$.

(C) Direct summands are stable.

(D) Let N_i be a summand of M_i for $i \in I$. Then $\bigoplus_{i \in I} N_i$ is stable in $\bigoplus_{i \in I} M_i$ if and only if each N_i is stable in M_i.

(E) Let A and B be submodules of M with $A \subseteq B \subseteq M$. If A is stable in M and B/A is stable in M/A, then B is stable in M.

(F) Let M have torsion free rank one and let $x \in M$. Then $[x]$ is stable in M.

(G) Let M be a T^*-module. Then M has a decomposition basis X such that, for each $Y \subseteq X$, $[Y]$ is stable in M.

Let M be a module and $\mu = \{\alpha_0, \alpha_1, \ldots\}$ be a proper indicator. The following were defined in [2]:

$\mu M = \{m \in M\colon H(m) \geqslant \mu\}$;

$\mu^* M = [\{m \in \mu M\colon \text{for infinitely many } i, h(p^i m) > \alpha_i\}]$.

Let A be a submodule of M. Then

$$M(p, A) = \{m \in M\colon \text{there is } k \geqslant 0 \text{ such that } p^k m \in A\}.$$

Let $\mu^*(M, A) = \mu M \cap (\mu^* M + M(p, A))$. Then $\mu M/\mu^*(M, A)$ is a vector space over R/pR if μ does not contain ∞, and is a free R-module if μ contains ∞. In either case, the rank $r(\mu M/\mu^*(M, A))$ is defined. We may now define the relative S-invariants.

Definition. Let M be a module, A a submodule and e an equivalence class of indicators. Then

$$S(e, M, A) = \sup_{\mu \in e} \{r(\mu M/\mu^*(M, A))\}.$$

These invariants are called *relative S-invariants*.

Let $\mu = \{\alpha_0, \alpha_1, \ldots\}$ be an indicator and $i \geqslant 0$.

Then μ_i is defined by $\mu_i = \{\alpha_i, \alpha_{i+1}, \ldots\}$.

Lemma 2.1. *The map* $\phi\colon \mu M/\mu^*(M, A) \to \mu_i M/\mu_i^*(M, A)$, *defined by*

$$\phi(x + \mu^*(M, A)) = p^i x + \mu_i^*(M, A),$$

is a monomorphism.

The proof is similar to Lemma 1 of [2].

3. **A Generalization of Warfield's Theorem**. The following theorem generalizes Theorem 5.2 of Warfield [5].

Theorem 3.1. *Let M and N be modules, with submodules A and B respec-*

tively, subject to the following conditions.

(i) *A is stable in M and B is stable in N.*

(ii) *M/A and N/B are T^*-modules.*

(iii) *$f(\alpha, M, A) = f(\alpha, N, B)$ for all ordinals α.*

(iv) *$S(e, M, A) = S(e, N, B)$ for all equivalence classes e of proper indicators.*

(v) *There is a height preserving isomorphism ψ: $A \to B$.*
Then ψ can be extended to an isomorphism ϕ: $M \to N$.

PROOF. By (G) of the previous section and (ii), M/A has a decomposition basis X such that X' is stable in M/A for every $X' \subseteq X$. Likewise N/B has a decomposition basis Y with the analogous property. We may assume, by the proof of Lemma 5.1 of [5], that X has the following property: Whenever $f(\alpha, M, A)$ is infinite, then $f(\alpha, M, A) = f(\alpha, M, A \oplus [X])$. Likewise, we may assume that whenever $f(\alpha, N, B)$ is infinite, then $f(\alpha, N, B) = f(\alpha, N, B \oplus [Y])$. It is also permissible to assume that there is a bijection β: $X \to Y$ such that $H(x) = H(\beta x)$ for all $x \in X$.

We interrupt the proof in order to show that a "one step extension" is possible.

LEMMA 3.2. *Let the situation of Theorem* 3.1 *be given and let $x \in X$. Then ψ can be extended to a height preserving isomorphism ζ: $A \oplus [x] \to B \oplus [\beta x]$ such that*:

(i) *$A \oplus [x]$ is stable in M and $B \oplus [\beta x]$ is stable in N.*

(ii) *$M/(A \oplus [x])$ and $N/(B \oplus [\beta x])$ are T^*-modules.*

(iii) *$f(\alpha, M, A \oplus [x]) = f(\alpha, N, B \oplus [\beta x])$ for all α.*

(iv) *$S(e, M, A \oplus [x]) = S(e, M, B \oplus [\beta x])$ for all e.*

PROOF. Since $[A, x]/A$ is free, $A \oplus [x]$ is really a direct sum, so the map ζ can be defined by $\zeta(a + rx) = \psi a + r\beta x$. ζ is trivially height preserving. Property (i) follows from (E), while (ii) is a known property of T^*-modules. For (iii), we use the following well-known formula.

$$f(\alpha, M, A \oplus [x]) = f(\alpha, M, A) - 1$$

if $f(\alpha, M, A)$ is finite, and there is r such that $h(rx) = \alpha$, but $h(prx) > \alpha + 1$.
$f(\alpha, M, A \oplus [x]) = f(\alpha, M, A)$ otherwise.

A standard argument shows the following, If e is a class of indicators, then

$$\{x \in X: H(x) \in e\} = S(e, M, A).$$

With this observation, (iv) is immediate.

CONCLUSION OF PROOF OF THEOREM 3.1. Let θ: $A \oplus [X] \to B \oplus [Y]$ be the unique isomorphism which extends ψ and β. θ is a height preserving isomorphism, $A \oplus [X]$ is nice in M and $B \oplus [Y]$ is nice in $[Y]$. $M/(A \oplus [X])$ and $N/(B \oplus [Y])$ are torsion T^*-modules. Because of the choice of X and Y and Lemma 3.2, the relative Ulm invariants are equal:

$$f(\alpha, M, A \oplus [X]) = f(\alpha, N, B \oplus [Y]).$$

By the analogue to Ulm's theorem for totally projective groups, (see Walker [3, Theorem 2.8]), θ can be extended to an isomorphism $\phi: M \to N$.

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