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CONDITIONS FOR COMMUTATIVE SEMIGROUPS TO HAVE NONTRIVIAL HOMOMORPHISMS INTO NONNEGATIVE (POSITIVE) REALS

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ABSTRACT. Let S be a commutative cancellative semigroup of finite rank (the free rank of the quotient group of S is finite). We give a necessary and sufficient condition for S to have a nontrivial homomorphism into the additive semigroup of nonnegative (positive) reals. We also give a counterexample in the case of infinite rank.

1. **Introduction and statement of Theorem**. Throughout this note "semigroup" means a commutative cancellative semigroup and its operation is written additively. $\mathbf{R}_0$ (resp. $\mathbf{R}_+$) denotes the additive semigroup of nonnegative (resp. positive) reals. It is a fundamental fact that any N-semigroup is homomorphic into $\mathbf{R}_+$ [4], [5]. What is a necessary and sufficient condition under which a semigroup has a nontrivial homomorphism into $\mathbf{R}_0$ (resp. $\mathbf{R}_+$)? This problem is treated in [3] in some special cases. In this note we solve the problem under a certain finiteness condition.

Let S be a semigroup and let $\tilde{S}$ be its quotient group. The rank of S means the free rank of $\tilde{S}$ and is denoted by rank S. $\mathbf{Z}_+$ denotes the set of all positive integers. The theorem we are going to prove is

THEOREM. *Let S be a commutative cancellative semigroup of finite rank. Then*:

(1) $\mathrm{Hom}(S, \mathbf{R}_0) \neq 0$ *if and only if S is not a group.*

(2) $\mathrm{Hom}(S, \mathbf{R}_+) \neq \varnothing$ *if and only if S satisfies the condition*

(*) *for any $a, b \in S$ there exists $n \in \mathbf{Z}_+$ such that $lna \nmid lb$*[1] *for all $l \in \mathbf{Z}_+$.*

In the last section we give an example; a semigroup S of infinite rank satisfying condition (*) but $\mathrm{Hom}(S, \mathbf{R}_0) = 0$ (a fortiori $\mathrm{Hom}(S, \mathbf{R}_+) = \varnothing$).

2. **Reduction**. Let $\mathbf{Z}$ be the ring of integers and let $\mathbf{Q}$ (resp. $\mathbf{R}$) be the rational (resp. real) number field. Assume rank $S = r < \infty$. Let $\tilde{S}$ be the quotient group of S and let ι_1: $S \to \tilde{S}$ be the injection. By the isomorphism $\tilde{S} \otimes_{\mathbf{Z}} \mathbf{Q} \xrightarrow{\sim} \mathbf{Q}^r$, we identify $\tilde{S} \otimes_{\mathbf{Z}} \mathbf{Q}$ and $\mathbf{Q}^r$. Let ι_2: $\tilde{S} \to \tilde{S} \otimes_{\mathbf{Z}} \mathbf{Q}$ be the

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[1] We write $a|b$ if either $a = b$ or $a + c = b$ for some $c \in S$, and otherwise we write $a \nmid b$.

canonical mapping. Let $\iota = \iota_1 \circ \iota_2$ and $C_S = \{a \in \mathbf{Q}^r | na \in \iota(S)$ for some $n \in \mathbf{Z}_+\}$.

LEMMA 1. (1) $\mathrm{Hom}(S, \mathbf{R}_0) \xrightarrow{\sim} \mathrm{Hom}(C_S, \mathbf{R}_0)$.
(2) $\mathrm{Hom}(S, \mathbf{R}_+) \xrightarrow{\sim} \mathrm{Hom}(C_S, \mathbf{R}_+)$.

PROOF. $f \in \mathrm{Hom}(S, \mathbf{R}_0)$ is extended uniquely to $\tilde{f} \in \mathrm{Hom}(\tilde{S}, \mathbf{R})$ in the natural way. $\tilde{f}$ is extended uniquely to $\bar{f} \in \mathrm{Hom}(\tilde{S} \otimes_{\mathbf{Z}} \mathbf{Q}, \mathbf{R})$ by $\bar{f}(s \otimes q) = \tilde{f}(s)q$. It is easy to see $\bar{f} \geqslant 0$ on C_S, and, moreover, $\bar{f} > 0$ on C_S if $f > 0$. This proves the lemma.

LEMMA 2. *S is a group if and only if C_S is a group.*

PROOF. If S is a group, $\iota(S)$ is a group. It is easy to prove that C_S is a group. Conversely, assume C_S is a group. Then for any $s \in S$ there exists $x \in C_S$ such that $s \otimes 1 + x = 0$. By the definition of C_S, $nx = s' \otimes 1 \in \iota(S)$ for some $n \in \mathbf{Z}_+$ and some $s' \in S$. Hence $(ns + s') \otimes 1 = 0$. Therefore there exists $m \in \mathbf{Z}_+$ such that $m(ns + s') = 0$. This shows $(mn - 1)s + ms'$ is the inverse element of s. Thus every element of S is invertible, and then S is a group.

The proof of the following lemma is routine and we omit it.

LEMMA 3. *S satisfies condition $(*)$ in §1 if and only if C_S satisfies the condition*

$$(**) \qquad \bigcap_{n=1}^{\infty} (nx + C_S) = \varnothing \quad \textit{for all } x \in C_S.$$

By the definition of C_S, C_S is a subsemigroup of $\mathbf{Q}^r$ satisfying that $nx \in C_S$ for some $n \in \mathbf{Z}_+$ implies $x \in C_S$. That is, C_S is a convex cone in $\mathbf{Q}^r$.

3. **Convex cones in $\mathbf{Q}^r$ and $\mathbf{R}^r$.** Let S be a convex cone in $\mathbf{Q}^r$. $\mathbf{Q}^r$ is considered as a subset of $\mathbf{R}^r$ naturally. Let $\bar{S}$ be the closure of S in $\mathbf{R}^r$ (we always consider the usual topology of $\mathbf{R}^r$). Then $\bar{S}$ is a closed convex cone in $\mathbf{R}^r$.

LEMMA 4. $\mathrm{Hom}(S, \mathbf{R}_0) \xrightarrow{\sim} \mathrm{Hom}(\bar{S}, \mathbf{R}_0)$.

PROOF. $f \in \mathrm{Hom}(S, \mathbf{R}_0)$ can be extended to $\tilde{f} \in \mathrm{Hom}(\tilde{S}, \mathbf{R})$, and $\tilde{f}$ can be extended to $F \in \mathrm{Hom}(\mathbf{Q}^r, \mathbf{R})$ since $\mathbf{R}$ is a divisible abelian group (see [2] for example). From the isomorphism $\mathrm{Hom}(\mathbf{Q}^r, \mathbf{R}) \xrightarrow{\sim} \mathbf{R}^r$, it follows that F (consequently f) is continuous. Therefore f is extended uniquely to $\bar{f} \in \mathrm{Hom}(\bar{S}, \mathbf{R}_0)$.

LEMMA 5. *Let A be a convex set of $\mathbf{Q}^r$ and let A' be the convex hull of A in $\mathbf{R}^r$. Then $A' \cap \mathbf{Q}^r = A$.*

LEMMA 6. *S is a group if and only if $\bar{S}$ is a group.*

PROOF. It is easy to see that $\bar{S}$ is a group if S is a group. Conversely, assume that $\bar{S}$ is a group. Since $\bar{S}$ is an $\mathbf{R}$-vector subspace of $\mathbf{R}^r$, we may assume

$\bar{S} = \mathbf{R}^r$. Since S is dense in $\mathbf{R}^r$, it follows that $S' = \mathbf{R}^r$. By Lemma 5, $S = S' \cap \mathbf{Q}^r = \mathbf{Q}^r$, thus S is a group.

$I(\bar{S})$ denotes the subgroup of $\bar{S}$ of all invertible elements of $\bar{S}$. Obviously $I(\bar{S})$ is an $\mathbf{R}$-vector subspace of $\mathbf{R}^r$.

LEMMA 7. $I(\bar{S}) \cap S = \{a \in S \mid \bigcap_{n=1}^{\infty}(na + S) \neq \emptyset\}$.

PROOF. Assume that $b \in \bigcap_{n=1}^{\infty}(na + S) \neq \emptyset$. Then $b - na \in S$ for all $n \in \mathbf{Z}_+$. Let $c_n = b/n - a$. Then $c_n \in S$. Since $c_n \to -a$ as $n \to \infty$, it follows that $-a \in \bar{S}$. This shows $a \in I(\bar{S}) \cap S$. Conversely let $a \in I(\bar{S}) \cap S$. Let L be the hyperplane through $-a$ perpendicular to the line through a and the origin. $L \cap S$ is convex and dense in $L \cap \bar{S}$. By Lemma 8, which we state below, there exists $b \in L \cap S$ such that $(b - na)/(n + 1) \in L \cap S$ for all $n \in \mathbf{Z}_+$. Hence $b - na \in S$ for all $n \in \mathbf{Z}_+$, that is, $b \in \bigcap_{n=1}^{\infty}(na + S) \neq \emptyset$. This completes the proof.

LEMMA 8. *Let A be a convex set of $\mathbf{Q}^r$ and let $\bar{A}$ be the closure of A in $\mathbf{R}^r$. Let $a \in \bar{A} \cap \mathbf{Q}^r$. Then there exists $b \in A$ such that every rational point of the open segment joining a and b is contained in A.*

PROOF. Let A' be the convex hull of A in $\mathbf{R}^r$. If A' has an inner point, we can choose an element b of A which is an inner point of A' since A is dense in A'. Then it is known that the open segment joining a and b is contained in A' (see [1, Proposition 16, p. 54]). Hence every rational point of the segment is contained in $A' \cap \mathbf{Q}^r = A$. If A' has no inner point, then A does not contain $r + 1$ points in general position. Therefore A is contained in a hyperplane of $\mathbf{Q}^r$. Thus the proof is reduced to the case of dimension $r - 1$. Repeating this, we may only prove the lemma in the case of $r = 1$, but in this case the assertion of the lemma is clear. The proof is completed.

4. **Proof of Theorem**. The necessity of the conditions of the Theorem is clear. By virtue of Lemmata 1, 2, 4 and 6, in order to prove (1) of the Theorem we may show for a convex cone S in $\mathbf{Q}^r$ that

(1′) if $\bar{S}$ is not a group, then $\mathrm{Hom}(\bar{S}, \mathbf{R}_0) \neq 0$.

By virtue of Lemmata 1, 3 and 7, in order to prove (2) of the Theorem we may show for a convex cone S in $\mathbf{Q}^r$ that

(2′) if $I(\bar{S}) \cap S = \emptyset$, then $\mathrm{Hom}(S, \mathbf{R}_+) \neq \emptyset$.

Let η: $\mathbf{R}^r \to \mathbf{R}^r/I(\bar{S})$ be the canonical surjection to the quotient space. Then $\eta(\bar{S})$ is a closed convex cone in $\mathbf{R}^r/I(\bar{S})$ satisfying $I(\eta(\bar{S})) = 0$. Then $\eta(\bar{S})$ is convex strictly at the origin, hence contained strictly in a half-space of $\mathbf{R}^r/I(\bar{S})$ except the origin. Therefore, there exists a functional φ of $\mathbf{R}^r/I(\bar{S})$ such that $\varphi(x) > 0$ for all $x \in \eta(\bar{S}) \setminus \{0\}$. Let $f = (\varphi \circ \eta)|_{\bar{S}}$, then $f \in \mathrm{Hom}(\bar{S}, \mathbf{R}_0)$ and $f(a) > 0$ for all $a \in \bar{S} \setminus I(\bar{S})$. If $\bar{S}$ is not a group, then $\bar{S} \setminus I(\bar{S}) \neq \emptyset$, hence $f \neq 0$. Therefore $\mathrm{Hom}(\bar{S}, \mathbf{R}_0) \neq 0$, this proves (1′). If $\bar{S}$ satisfies $I(\bar{S}) \cap S = \emptyset$, then $f|_S > 0$. Therefore $\mathrm{Hom}(S, \mathbf{R}_+) \neq \emptyset$; this proves (2′). The proof of the Theorem is complete.

5. **Counterexample in the case of infinite rank.** In the case of infinite rank, condition (*) in the Theorem does not imply $\mathrm{Hom}(S, \mathbf{R}_0) \neq 0$ as the following example S shows. S is constructed as a subsemigroup of the free group $\bigoplus_1^\infty \mathbf{Z}$ of countable rank. Let T_n be an $n \times n$-matrix:

$$T_n = \begin{bmatrix} \binom{n-1}{n-1} & \binom{n-1}{n-2} & \cdots & \binom{n-1}{0} \\ & \binom{n-2}{n-2} & \cdots & \binom{n-2}{0} \\ & & & \vdots \\ & & \ddots & \vdots \\ & 0 & & \binom{0}{0} \end{bmatrix},$$

where $\binom{\cdot}{\cdot}$ denotes the binomial coefficients.

For $x = {}^t(x_1, x_2, \ldots, x_i, \ldots) \in \bigoplus_1^\infty \mathbf{Z}$ or for $x = {}^t(x_1, x_2, \ldots, x_m) \in \bigoplus_1^m \mathbf{Z}$ with $m \geqslant n$, we define $\tau_n(x) = {}^t(x_1, x_2, \ldots, x_n)$ and $\pi_n(x) = x_n$. Set $S(n) = \{x \in \bigoplus_1^\infty \mathbf{Z} | T_n\tau_n(x) > 0$, and $\pi_m(x) = 0$ for $m > n\}$, where $T_n\tau_n(x) > 0$ means that all components of $T_n\tau_n(x)$ are positive. Now we define $S = \bigcup_{n=1}^\infty S(n) \subset \bigoplus_1^\infty \mathbf{Z}$.

S is a semigroup: We define $\bar{S}(n) = \{x \in \bigoplus_1^\infty \mathbf{Z} | T_n\tau_n(x) \geqslant 0$, and $\pi_m(x) = 0$ for $m > n\}$. By a simple calculation using the formula

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-2}{r-1} + \cdots + \binom{r-1}{r-1},$$

we find $\bar{S}(n) \subset \bar{S}(m)$ if $m > n$. Hence $S(m) + S(n) \subset S(m) + \bar{S}(m) \subset S(m)$. This shows that S is closed with respect to addition, so S is a semigroup.

S satisfies condition (*): Let $a \in S(p)$ and $b \in S(q)$. If $p > q$, then clearly $l(b - a) \notin S$ for all $l \in \mathbf{Z}_+$. If $p \leqslant q$, then $\pi_p(a) > 0$, and $\pi_{p'}(a) = 0$ for $p' > p$. Hence $\pi_p(T_q\tau_q(a)) = \pi_p(a) > 0$. If we choose $n \in \mathbf{Z}_+$ such that $n > \pi_p(T_q\tau_q(b))/\pi_p(a)$, then

$$\pi_p\big(T_q\tau_q(l(b - na))\big) = l\big(\pi_p(T_q\tau_q(b)) - n\pi_p(a)\big) < 0$$

for all $l \in \mathbf{Z}_+$. This implies $l(b - na) \notin S$ for all $l \in \mathbf{Z}_+$. Thus we see in either case that S satisfies condition (*).

$\mathrm{Hom}(S, \mathbf{R}_0) = 0$: We define $\bar{S} = \bigcup_{n=1}^\infty \bar{S}(n)$. Since $f \in \mathrm{Hom}(S(n), \mathbf{R}_0)$ is extended uniquely to $\bar{f} \in \mathrm{Hom}(\bar{S}(n), \mathbf{R}_0)$, $\varphi \in \mathrm{Hom}(S, \mathbf{R}_0)$ is extended uniquely to $\bar{\varphi} \in \mathrm{Hom}(\bar{S}, \mathbf{R}_0)$. Hence it suffices to prove $\mathrm{Hom}(\bar{S}, \mathbf{R}_0) = 0$. Let $n_0 \in \mathbf{Z}_+$ and let $x \in \bigoplus_1^\infty \mathbf{Z}$ satisfy that $\pi_{n_0}(x) > 0$ and $\pi_n(x) = 0$ for $n > n_0$. Then we easily have $T_m\tau_m(x) \geqslant 0$ for a sufficiently large m, hence $x \in \bar{S}(m) \subset \bar{S}$. Therefore, for any $a \in \bar{S}$ (say $a \in \bar{S}(p)$) there exists $b \in S$ (any $b \in S(p + 1)$ is available) such that $la|b$ in $\bar{S}$ for all $l \in \mathbf{Z}_+$. From this it follows immediately that $\mathrm{Hom}(\bar{S}, \mathbf{R}_0) = 0$.

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