ON AUTOMORPHISMS OF L.C. GROUPS

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Abstract. The left regular representation \( \lambda \) of a locally compact group \( G \) generates a \( W^* \)-algebra \( \mathfrak{A}(\lambda) \), and each topological automorphism \( \varphi \) of \( G \) has a natural extension to an automorphism \( \varphi_{\mathfrak{A}} \) of \( \mathfrak{A}(\lambda) \). It is proved that an automorphism \( \beta \) of \( \mathfrak{A}(\lambda) \) is of the form \( \beta = \varphi_{\mathfrak{A}} \) for \( \varphi \in \text{Aut}(G) \) iff \( \beta \) leaves a certain cone in \( \mathfrak{A}(\lambda) \) invariant.

Recently a number of important results concerning automorphisms of \( W^* \)-algebras have been obtained (see e.g. [2], [4]), and in some cases these results have analogues in automorphisms of locally compact groups, or can be applied directly to yield facts about group automorphisms [6]. Let \( \lambda \) denote the left regular representation of a locally compact group \( G \) \((\lambda(x), x \in G\), operates on an \( L^2(G) \)-function by left translation by \( x^{-1} \)) and \( \mathfrak{R}(G) \) be the double commutant of \( \lambda(G) = \{\lambda(x) : x \in G\} \), or the \( W^* \)-algebra generated by \( \lambda(G) \). There is a natural imbedding of \( \text{Aut}(G) \hookrightarrow \text{Aut} \mathfrak{R}(G) \): to \( \varphi \in \text{Aut}(G) \) there corresponds a unique \( \varphi_{\mathfrak{R}} \in \text{Aut} \mathfrak{R}(G) \) satisfying \( \varphi_{\mathfrak{R}}(\lambda(x)) = \lambda(\varphi(x)) \). In this note we characterize those automorphisms of \( \mathfrak{R}(G) \) which come from automorphisms of \( G \) (via the imbedding) as the set of automorphisms of \( \mathfrak{R}(G) \) which leave a certain cone fixed. Also, a connection between automorphisms of \( \mathfrak{R}(G) \) and the measure algebra \( \mathfrak{M}(G) \) is mentioned.

If \( A \) is a Banach \(*\)-algebra, we denote by \( A^+ \) the positive cone in the dual \( A' \) given by \( \{f \in A' : f(a^*a) \geq 0, a \in A\} \). There is also a cone \( A^* = \{a \in A : f(a) \geq 0, f \in A^+\} \) in \( A \). A ray in \( A^+ \) is a set of the form \( R^+f, f \in A^+ \) \((R^+ = \{r \in R : r > 0\}) \). A linear functional \( f \in A^+ \) is said to lie on an extreme ray if \( f = g + h, g, h \in A^{++} \), implies \( g, h \in R^+f \). The following proposition is an extension of a well-known theorem of Kelley and Vaught [5].

Proposition. Let \( A \) be a commutative Banach \(*\)-algebra with continuous involution, and \( B \subset A \) a dense subalgebra. Suppose \( \{e_i\} \subset B \cap A^* \) is a (not necessarily bounded) approximate identity for \( B \). Let \( f \in A^{++} \); then \( f \) lies on an extreme ray if and only if \( \theta f \) is multiplicative for some \( \theta, 0 \neq \theta \in R^+ \).

Proof. Let \( f \in A^{++} \) lie on an extreme ray and choose \( b \in B \) such that
\[ \|b^*b\| < 1 \text{ and } f(b^*b) > 0. \] Define \( g \in A^+ \) by \( g(y) = f(yb^*b) \). Clearly \( g \) is positive since \( g(y^*y) = f((yb)^*(yb)) > 0. \) Likewise \( (f - g)(y^*y) = f(z^*z) \), where \( z = \sum_{n=0}^{\infty} \frac{(1/2)^n}{n!}(-b^*b)^n \). (Observe that if we adjoined an identity to \( A \) in the canonical way we could write
\[
 z = y\sqrt{1 - b^*b} = y\sum_{n=0}^{\infty} \left(\frac{1/2}{n!}\right)(-b^*b)^n.
\]
Since we can write \( f = g + (f - g) \) with \( g, (f - g) \in A^+ \), it follows from the assumption on \( f \) that \( g = \mu f \), for some \( \mu, 0 < \mu < \infty \). But \( g(e_i) = f(e_i b^*b) \neq 0 \) for sufficiently large \( i \), so \( \mu > 0 \). This implies \( \lim f(e_i) \) exists, and, in fact,
\[
 \lim f(e_i) = \mu^{-1}\lim g(e_i) = \mu^{-1}\lim f(e_i b^*b) = \mu^{-1}f(b^*b) < \infty.
\]
Let
\[
 \theta = \mu/f(b^*b) = 1/\lim f(e_i).
\]
Note that \( \theta \) does not depend on \( b \). Then \( \theta g = \mu\theta f \) and \( \mu = \theta f(b^*b) \). Thus
\[
 \theta g(y) = \theta f(b^*b)\theta f(y), \quad y \in A,
\]
or
\[
 \theta f(b^*b) = \theta f(b^*b)\theta f(y), \quad y \in A.
\]
Observe now that this last equation is valid if in place of \( b^*b \) we have \( a^*a \), \( a \in A \), where \( f(a^*a) = 0 \). For in that case \( |f(a^*ay)|^2 < f(a^*a)f((ay)^*(ay)) = 0. \) Since any \( x \in B^2 \) can be written as a linear combination of elements of the form \( b^*b, \|b^*b\| < 1 \), we have \( \theta f(xy) = \theta f(x)\theta f(y), x \in B^2, y \in A \).

Finally, using that \( B^2 \subset A \) is dense along with the continuity of \( f \), we have that \( \theta f(xy) = \theta f(x)\theta f(y) \) holds for all \( x, y \in A \).

The converse is easy. For let \( f \in A^+ \) be multiplicative and suppose \( f = g + h, g, h \in A^+ \). Let \( N_f \) (resp., \( N_g \)) be the null space of \( f \) (resp., \( g \)). If \( x \in N_f \), then \( f(x^*x) = f(x^*)f(x) = 0 \), hence \( g(x^*x) = 0 \). But then \( |g(x)|^2 < g(x^*x) = 0 \), so \( N_g \subset N_f \). This means \( g = \mu f \) for some \( \mu, 0 < \mu < \infty \), and, consequently, \( f \) lies on an extreme ray. \( \square \)

We apply the foregoing proposition by taking \( A \) to be \( A(G) \), the Fourier algebra of a locally compact group \( G \). Recall that \( A(G) \) is a Banach* - algebra of continuous functions on \( G \) vanishing at infinity under pointwise multiplication with complex conjugation as involution. If \( C_c(G) \) denotes the continuous functions with compact support on \( G \), then \( B = C_c(G) \cap A(G) \) is dense in \( A(G) \). Given a compact set \( k \subset G \) there is a function \( a_k \in B \), \( 0 < a_k \leq 1 \), \( a_k[k = 1] \). If for each compact set \( k \subset G \) we choose such an \( a_k \) and order the \( k \)'s by inclusion and set \( e_k = a_k^2 = a_k^*a_k \), then \( \{e_k\} \subset B \cap A^+ \) constitutes an (unbounded) approximate identity for \( B \), so the hypotheses of the proposition are satisfied.

Now \( A(G)' = \mathbb{R}(G) \), and \( \Delta(A(G)) \), the spectrum of \( A(G) \), is identified with \( G \), or, more properly, \( \lambda(G) \) (acting by pointwise evaluation) \( [3, 3.34] \). Let \( P = A(G)^+ \). The cone \( P \) is not to be confused with the cone of positive
operators in $\mathcal{R}(G)$: indeed, $\lambda(x) (x \in G)$ is clearly in $P$, but $\lambda(x)$ is not even hermitian if $x \neq x^{-1}$. Applying the above proposition, we find that the set of extreme rays of $P$ is precisely $\{ R^* \lambda(x): x \in G \}$.

By an automorphism of a $W^\ast$ algebra we mean, of course, a $\ast$-automorphism, and this will automatically be norm-preserving.

**Corollary.** An automorphism $\beta$ of $\mathcal{R}(G)$ comes from an automorphism of $G$ if and only if $\beta P = P$.

**Proof.** $P$ has a metrizable topology and so is well capped [1, 30.19]; hence by the Choquet theory $P$ is the closed convex hull of its extreme rays. If $\beta = \tilde{a}$, $a \in \text{Aut}(G)$, $\beta$ maps the set of extreme rays of $P$ onto itself, thus $\beta P = P$. Suppose, conversely, that $\beta P = P$. If $\beta \lambda(x) = S + T$, $S, T \in P$, then $\lambda(x) = \beta^{-1} S + \beta^{-1} T$. Thus $\beta^{-1} S = \mu \lambda(x)$, some $0 < \mu < \infty$, or $S = \mu \beta \lambda(x)$, which means $\beta \lambda(x)$ lies on an extreme ray. Since $\| \beta \lambda(x) \| = 1$, we must have $\beta \lambda(x) = \lambda(y)$, for some $y \in G$. Setting $y = \alpha(x)$ we obtain a map $\alpha: G \to G$. An easy argument now shows $\alpha \in \text{Aut}(G)$. □

Instead of looking at the extreme rays of a certain cone in $\mathcal{R}(G)$ we could, of course, consider the extreme points of the unit ball in $\mathcal{R}(G)$. But any $\beta \in \text{Aut} \mathcal{R}(G)$ must map $\text{Ext}(\mathcal{R}(G)_1)$ onto itself, so in place of $\mathcal{R}(G)_1$ we might take $\mathcal{M}(G)_1$, where $\mathcal{M}(G)$ is the measure algebra. Now $A(G) \subset C_0(G)$, the continuous functions on $G$ vanishing at infinity, so we can view $\mathcal{M}(G) = C_0(G)'$ as a subalgebra of $\mathcal{R}(G)$. If $\mathcal{M}(G)$ is given its own norm (and not the norm it inherits from $\mathcal{R}(G)$), then $\text{Ext}(\mathcal{M}(G)_1) = \{ e^{\delta_x}: \theta \in R, x \in G \}$, $\delta_x$ being the point mass at $x$. Indeed, if $\mu \in \mathcal{M}(G)_1$, $\| \mu \| = 1$, is nonatomic, then there are nonzero $\mu_1, \mu_2 \in \mathcal{M}(G)$, support($\mu_i$) $\subset$ support($\mu$), $i = 1, 2$, satisfying support($\mu_1$) $\cap$ support($\mu_2$) is a $|\mu|$-null set, and $\mu = \mu_1 + \mu_2$ with $\| \mu_1 \| + \| \mu_2 \| = \| \mu \| = 1$. Setting $V_i = \| \mu_i \|_1^{-1} \mu_i$, $i = 1, 2$, we have $\mu = \| \mu_1 \|_1 V_1 + \| \mu_2 \|_2 V_2$, so $\mu$ is not extreme. On the other hand, $\mu e^{\delta_x} \in \text{Ext}(\mathcal{M}(G)_1)$. In fact, viewing $\mathcal{M}(G) \subset \mathcal{R}(G)$, $\mu e^{\delta_x}$ belongs to the larger unit ball $\mathcal{R}(G)_1$, and $\mu e^{\delta_x} \in \text{Ext}(\mathcal{R}(G)_1)$ by [7, 1.6.4].

**Proposition.** $\beta \in \text{Aut} \mathcal{R}(G)$ restricts to an isometric automorphism of $\mathcal{M}(G)$ if and only if $\beta = \tilde{a} \circ \gamma$, where $\beta \in \text{Aut}(G)$ and $\gamma$ is a group character.

**Proof.** Consider first that a group character acts as a $\ast$-automorphism of the group algebra $L^1(G)$: defining $(\gamma f)(x) = \gamma(x)f(x), f \in L^1(G)$, we have that $(\gamma f) \ast (\gamma g) = \gamma(f \ast g), f, g \in L^1(G)$, and $(\gamma f)^\ast = \gamma f^\ast$, where $f^\ast(x) = \Delta(x)^{-1} f(x^{-1})^{-1}$ is the involution in $L^1(G)$. It is clear that $\gamma$ extends to an isometric $\ast$-automorphism of the measure algebra; for $\mu \in \mathcal{M}(G)$, we have $d(\gamma \mu)(x) = \gamma(x) d\mu(x)$. Let $\gamma$ act on $\mathcal{R}(G)$ by defining $(\gamma T) g = \gamma(T(\gamma g)), T \in \mathcal{R}(G), g \in L^2(G)$, where $\gamma$ acts by pointwise multiplication on $L^2(G)$; i.e., $\gamma T = \gamma T\gamma$. To see this define a $\ast$-automorphism of $\mathcal{R}(G)$, first note for $T \in \mathcal{R}(G), g, h \in L^2(G)$ that

$$((\gamma T)^\ast g, h) = (g, \gamma T h) = (g, \gamma T \gamma h) = (\gamma g, T \gamma h) = (T^\ast \gamma g, \gamma h) = (\gamma T^\ast \gamma g, h) = (\gamma T^\ast g, h).$$
To show $\hat{\gamma}$ is multiplicative, observe, if $f \in L^1(G)$, $\hat{\gamma}\lambda(f) = \lambda(\gamma f)$. (We use the same notation for the left regular representation of $L^1(G)$, which acts by left convolution on $L^2(G)$, as we do for the left regular representation of $G$, since the former is just the Bochner-integrated form of the latter.) So for $f, g \in L^1(G)$,

$$
\hat{\gamma}(\lambda(f)\lambda(g)) = \hat{\gamma}(f \ast g) = \lambda(\gamma(f \ast g))
$$

$$
= \lambda((\gamma f) \ast (\gamma g)) = \lambda(\gamma f)\lambda(\gamma g) = \hat{\gamma}\lambda(f)\hat{\gamma}\lambda(g).
$$

Then use that $\{\lambda(f): f \in L^1(G)\} \subset \mathfrak{H}(G)$ is strongly dense and the fact that multiplication in $\mathfrak{H}(G)$ is jointly strongly continuous on bounded subsets to see $\hat{\gamma}$ is multiplicative on $\mathfrak{H}(G)$.

Suppose now $\beta \in \text{Aut} \mathfrak{H}(G)$ restricts to an isometric automorphism of $\mathfrak{H}(G)$. Then $\beta$ maps the unit ball of $\mathfrak{H}(G)$ onto itself, hence $\text{Ext}(\mathfrak{H}(G))$ is mapped onto itself. Thus $\beta \delta_x = e^{i\theta} \delta_{\alpha(x)}$, for some $\theta \in R$, and $\alpha: G \to G$. If we set $|\beta| \delta_x = \delta_{\alpha(x)}$, it is clear that $|\beta|$ is multiplicative on $\text{Ext}(\mathfrak{H}(G))$, hence $\alpha \in \text{Aut}(G)$. Set $\beta \delta_x = \gamma(x) \delta_{\alpha(x)}$. A simple argument shows $\gamma$ is multiplicative, hence is a group character. Thus $\beta$ agrees with $\tilde{\alpha} \circ \tilde{\gamma}$ on $\lambda(G)$, and since $\lambda(G)$ generates $\mathfrak{H}(G)$, we must have $\beta = \tilde{\alpha} \circ \tilde{\gamma}$.

**References**