ON AUTOMORPHISMS OF L.C. GROUPS

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Abstract. The left regular representation \( \lambda \) of a locally compact group \( G \) generates a \( W^* \)-algebra \( \mathcal{A}(\lambda) \), and each topological automorphism \( \alpha \) of \( G \) has a natural extension to an automorphism \( \tilde{\alpha} \) of \( \mathcal{A}(\lambda) \). It is proved that an automorphism \( \beta \) of \( \mathcal{A}(\lambda) \) is of the form \( \beta = \tilde{\alpha} \) for \( \alpha \in \text{Aut}(G) \) iff \( \beta \) leaves a certain cone in \( \mathcal{A}(\lambda) \) invariant.

Recently a number of important results concerning automorphisms of \( W^* \)-algebras have been obtained (see e.g. [2], [4]), and in some cases these results have analogues in automorphisms of locally compact groups, or can be applied directly to yield facts about group automorphisms [6]. Let \( \lambda \) denote the left regular representation of a locally compact group \( G \) (\( \lambda(x), x \in G \), operates on an \( L^2(G) \)-function by left translation by \( x^{-1} \)) and \( \mathcal{A}(G) \) be the double commutant of \( \lambda(G) = \{ \lambda(x): x \in G \} \), or the \( W^* \)-algebra generated by \( \lambda(G) \). There is a natural imbedding of \( \text{Aut}(G) \hookrightarrow \text{Aut} \mathcal{A}(G) \): to \( \alpha \in \text{Aut}(G) \) there corresponds a unique \( \tilde{\alpha} \in \text{Aut} \mathcal{A}(G) \) satisfying \( \tilde{\alpha}\lambda(x) = \lambda(\alpha(x)) \). In this note we characterize those automorphisms of \( \mathcal{A}(G) \) which come from automorphisms of \( G \) (via the imbedding) as the set of automorphisms of \( \mathcal{A}(G) \) which leave a certain cone fixed. Also, a connection between automorphisms of \( \mathcal{A}(G) \) and the measure algebra \( \mathcal{M}(G) \) is mentioned.

If \( A \) is a Banach \( * \)-algebra, we denote by \( A^+ \) the positive cone in the dual \( A^* \) given by \( \{ f \in A^*: f(a^*a) \geq 0, a \in A \} \). There is also a cone \( A^+ = \{ a \in A: f(a) \geq 0, f \in A^+ \} \) in \( A \). A ray in \( A^+ \) is a set of the form \( R^+f, f \in A^+ \) \( (R^+ = \{ r \in R: r \geq 0 \}) \). A linear functional \( f \in A^+ \) is said to lie on an extreme ray if \( f = g + h, g, h \in A^+ \), implies \( g, h \in R^+f \). The following proposition is an extension of a well-known theorem of Kelley and Vaught [5].

**Proposition.** Let \( A \) be a commutative Banach \( * \)-algebra with continuous involution, and \( B \subset A \) a dense subalgebra. Suppose \( \{ e_i \} \subset B \cap A^+ \) is a (not necessarily bounded) approximate identity for \( B \). Let \( f \in A^+ \); then \( f \) lies on an extreme ray if and only if \( \theta f \) is multiplicative for some \( \theta, 0 \neq \theta \in R^+ \).

**Proof.** Let \( f \in A^+ \) lie on an extreme ray and choose \( b \in B \) such that...
\[ \|b*b\| < 1 \text{ and } f(b*b) > 0. \] Define \( g \in A^+ \) by \( g(y) = f(yb*b) \). Clearly \( g \) is positive since \( g(y*y) = f((yb)*(yb)) > 0 \). Likewise \( (f - g)(y*y) = f(z*z) \), where \( z = \sum_{n=0}^{\infty} (\frac{1}{n^2})y(-b*b)^n \). (Observe that if we adjoined an identity to \( A \) in the canonical way we could write
\[
z = y \sqrt{1 - b*b} = y \sum_{n=0}^{\infty} \left( \frac{1}{n^2} \right)(-b*b)^n.
\]
Since we can write \( f = g + (f - g) \) with \( g, (f - g) \in A^+ \), it follows from the assumption on \( f \) that \( g = \mu f \), for some \( \mu \), \( 0 < \mu < \infty \). But \( g(e_i) = f(e_i b*b) \neq 0 \) for sufficiently large \( i \), so \( \mu > 0 \). This implies \( \lim f(e_i) \) exists, and, in fact,
\[
\lim f(e_i) = \mu^{-1} \lim g(e_i) = \mu^{-1} \lim f(e_i b*b) = \mu^{-1} f(b*b) < \infty.
\]
Let
\[
\theta = \frac{1}{f(b*b)} = 1/\lim f(e_i).
\]
Note that \( \theta \) does not depend on \( b \). Then \( \theta g = \mu \theta f \) and \( \mu = \theta f(b*b) \). Thus
\[
\theta g(y) = \theta f(b*b) \theta f(y), \quad y \in A,
\]
or
\[
\theta f(b*by) = \theta f(b*b) \theta f(y), \quad y \in A.
\]
Observe now that this last equation is valid if in place of \( b*b \) we have \( a*a \), \( a \in A \), where \( f(a*a) = 0 \). For in that case \( |f(a*ay)^2| < f(a*a)f((ay)*(ay)) = 0 \). Since any \( x \in B^2 \) can be written as a linear combination of elements of the form \( b*b \), \( \|b*b\| < 1 \), we have \( \theta f(xy) = \theta f(x) \theta f(y), x \in B^2, y \in A \). Finally, using that \( B^2 \subset A \) is dense along with the continuity of \( f \), we have that \( \theta f(xy) = \theta f(x) \theta f(y) \) holds for all \( x, y \in A \).

The converse is easy. For let \( f \in A^+ \) be multiplicative and suppose \( f = g + h \), \( g, h \in A^+ \). Let \( N_f \) (resp., \( N_g \)) be the null space of \( f \) (resp., \( g \)). If \( x \in N_g \), then \( f(x^*x) = f(x)^*f(x) = 0 \), hence \( g(x^*x) = 0 \). But then \( |g(x)|^2 < g(x^*x) = 0 \), so \( N_g \subset N_f \). This means \( g = \mu f \) for some \( \mu \), \( 0 < \mu < \infty \), and, consequently, \( f \) lies on an extreme ray. □

We apply the foregoing proposition by taking \( A \) to be \( A(G) \), the Fourier algebra of a locally compact group \( G \). Recall that \( A(G) \) is a Banach*-algebra of continuous functions on \( G \) vanishing at infinity under pointwise multiplication with complex conjugation as involution. If \( C_0(G) \) denotes the continuous functions with compact support on \( G \), then \( B = C_0(G) \cap A(G) \) is dense in \( A(G) \). Given a compact set \( k \subset G \) there is a function \( a_k \in B \), \( 0 < a_k \leq 1 \). If for each compact set \( k \subset G \) we choose such an \( a_k \), order the \( k \)'s by inclusion and set \( e_k = a_k^2 = a_k* a_k \), then \( \{e_k\} \subset B \cap A^+ \) constitutes an (unbounded) approximate identity for \( B \), so the hypotheses of the proposition are satisfied.

Now \( A(G)' = \mathbb{R}(G) \), and \( \Delta(A(G)) \), the spectrum of \( A(G) \), is identified with \( G \), or, more properly, \( \lambda(G) \) (acting by pointwise evaluation) \([3, 3.34]\). Let \( P = A(G)^+ \). The cone \( P \) is not to be confused with the cone of positive
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operators in \( \mathcal{R}(G) \): indeed, \( \lambda(x) \ (x \in G) \) is clearly in \( P \), but \( \lambda(x) \) is not even hermitian if \( x \neq x^{-1} \). Applying the above proposition, we find that the set of extreme rays of \( P \) is precisely \( \{ R^* \lambda(x) : x \in G \} \).

By an automorphism of a \( \mathcal{W}^* \) algebra we mean, of course, a *-automorphism, and this will automatically be norm-preserving.

**Corollary.** An automorphism \( \beta \) of \( \mathcal{R}(G) \) comes from an automorphism of \( G \) if and only if \( \beta P = P \).

**Proof.** \( P \) has a metrizable topology and so is well capped \([1, 30.19]\); hence by the Choquet theory \( P \) is the closed convex hull of its extreme rays. If \( \beta = \alpha, \alpha \in \text{Aut}(G) \), \( \beta \) maps the set of extreme rays of \( P \) onto itself, thus \( \beta P = P \). Suppose, conversely, that \( \beta P = P \). If \( \beta \lambda(x) = S + T, S, T \in P \), then \( \lambda(x) = \beta^{-1} S + \beta^{-1} T \). Thus \( \beta^{-1} S = \mu \lambda(x) \), some \( 0 < \mu < \infty \), or \( S = \mu \beta \lambda(x) \), which means \( \beta \lambda(x) \) lies on an extreme ray. Since \( \| \beta \lambda(x) \| = 1 \), we must have \( \beta \lambda(x) = \lambda(y) \), for some \( y \in G \). Setting \( y = \alpha(x) \) we obtain a map \( \alpha: G \to G \). An easy argument now shows \( \alpha \in \text{Aut}(G) \). □

Instead of looking at the extreme rays of a certain cone in \( \mathcal{R}(G) \) we could, of course, consider the extreme points of the unit ball in \( \mathcal{R}(G), \text{Ext}(\mathcal{R}(G)) \). But any \( \beta \in \text{Aut}(G) \) must map \( \text{Ext}(\mathcal{R}(G)) \) onto itself, so in place of \( \mathcal{R}(G)_1 \) we might take \( \mathcal{M}(G)_1 \), where \( \mathcal{M}(G) \) is the measure algebra. Now \( A(G) \subseteq C_0(G) \), the continuous functions on \( G \) vanishing at infinity, so we can view \( \mathcal{M}(G) = C_0(G) \) as a subalgebra of \( \mathcal{R}(G) \). If \( \mathcal{M}(G) \) is given its own norm (and not the norm it inherits from \( \mathcal{R}(G) \)), then \( \text{Ext}(\mathcal{M}(G)_1) = \{ e^{\# \delta_x} : \theta \in R, x \in G \} \), \( \delta_x \) being the point mass at \( x \). Indeed, if \( \mu \in \mathcal{M}(G)_1 \), \( \| \mu \| = 1 \), is nonatomic, then there are nonzero \( \mu_1, \mu_2 \in \mathcal{M}(G) \), support(\( \mu_i \)) \( \subseteq \) support(\( \mu \)), \( i = 1, 2 \), satisfying support(\( \mu_1 \)) \( \cap \) support(\( \mu_2 \)) is a \( \| \mu \| \)-null set, and \( \mu = \mu_1 + \mu_2 \) with \( \| \mu_1 \| + \| \mu_2 \| = \| \mu \| = 1 \). Setting \( V_i = \| \mu_i \|^{-1} \mu_i, i = 1, 2 \), we have \( \mu = \| \mu_1 \| V_1 + || \mu_2 || V_2 \), so \( \mu \) is not extreme. On the other hand, \( e^{\# \delta_x} \in \text{Ext}(\mathcal{M}(G)_1) \). In fact, viewing \( \mathcal{M}(G) \subseteq \mathcal{R}(G) \), \( e^{\# \delta_x} \) belongs to the larger unit ball \( \mathcal{R}(G)_1 \), and \( e^{\# \delta_x} \in \text{Ext}(\mathcal{R}(G)_1) \) by \([7, 1.6.4]\).

**Proposition.** \( \beta \in \text{Aut} \mathcal{R}(G) \) restricts to an isometric automorphism of \( \mathcal{M}(G) \) if and only if \( \beta = \alpha \circ \gamma \), where \( \beta \in \text{Aut}(G) \) and \( \gamma \) is a group character.

**Proof.** Consider first that a group character acts as a *-automorphism of the group algebra \( L^1(G) \): defining \((\gamma f)(x) = \gamma(x)f(x), f \in L^1(G)\), we have that \((\gamma f) \ast (\gamma g) = \gamma(f \ast g), f, g \in L^1(G)\), and \((\gamma f)^* = \gamma f^* \), where \( f^*(x) = \Delta(x)^{-1} f(x^{-1})^{-1} \) is the involution in \( L^1(G) \). It is clear that \( \gamma \) extends to an isometric *-automorphism of the measure algebra; for \( \mu \in \mathcal{M}(G) \), we have \( d(\gamma \mu)(x) = \gamma(x) d\mu(x) \). Let \( \gamma \) act on \( \mathcal{R}(G) \) by defining \((\gamma T)g = \gamma(T(\gamma g)), T \in \mathcal{R}(G), g \in L^2(G)\), where \( \gamma \) acts by pointwise multiplication on \( L^2(G) \); i.e., \( \gamma T = \gamma T \gamma \). To see this define a *-automorphism of \( \mathcal{R}(G) \), first note for \( T \in \mathcal{R}(G), g, h \in L^2(G) \) that

\[
((\gamma T)^* g, h) = (g, \gamma Th) = (g, \gamma T \gamma h) = (\gamma g, T \gamma h)
\]

\[
= (T^* \gamma g, \gamma h) = (\gamma T^* \gamma g, h) = (\gamma T^* g, h).
\]
To show $\hat{\gamma}$ is multiplicative, observe, if $f \in L^1(G)$, $\hat{\gamma} \lambda(f) = \lambda(\gamma f)$. (We use the same notation for the left regular representation of $L^1(G)$, which acts by left convolution on $L^2(G)$, as we do for the left regular representation of $G$, since the former is just the Bochner-integrated form of the latter.) So for $f, g \in L^1(G)$,

$$\hat{\gamma} (\lambda(f) \lambda(g)) = \hat{\gamma} (f * g) = \lambda(\gamma (f * g))$$

$$= \lambda(\gamma(f) * (\gamma g)) = \lambda(\gamma f) \lambda(\gamma g) = \hat{\gamma} \lambda(f) \hat{\gamma} \lambda(g).$$

Then use that $\{\lambda(f) : f \in L^1(G)\} \subset \mathbb{R}(G)$ is strongly dense and the fact that multiplication in $\mathbb{R}(G)$ is jointly strongly continuous on bounded subsets to see $\hat{\gamma}$ is multiplicative on $\mathbb{R}(G)$.

Suppose now $\beta \in \text{Aut} \mathbb{R}(G)$ restricts to an isometric automorphism of $\mathbb{R}(G)$. Then $\beta$ maps the unit ball of $\mathbb{R}(G)$ onto itself, hence $\text{Ext}(\mathbb{R}(G))$ is mapped onto itself. Thus $\beta \delta_x = e^{i\theta} \delta_{a(x)}$, for some $\theta \in \mathbb{R}$, and $a : G \to G$. If we set $|\beta| \delta_x = \delta_{a(x)}$, it is clear that $|\beta|$ is multiplicative on $\text{Ext}(\mathbb{R}(G))$, hence $a \in \text{Aut}(G)$. Set $\beta \delta_x = \gamma(x) \delta_{a(x)}$. A simple argument shows $\gamma$ is multiplicative, hence is a group character. Thus $\beta$ agrees with $\bar{a} \circ \hat{\gamma}$ on $\lambda(G)$, and since $\lambda(G)$ generates $\mathbb{R}(G)$, we must have $\beta = \bar{a} \circ \hat{\gamma}$.

REFERENCES


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