PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 65, Number 2, August 1977

AN INDIVIDUAL ERGODIC THEOREM

RYOTARO SATO

ABSTRACT. An individual ergodic theorem is proved for a linear operator T on L_1 of a finite measure space which satisfies certain norm conditions.

Derriennic and Lin [5] showed by an example that given an $\varepsilon > 0$ there exists a positive linear operator T on L_1 of a finite measure space, with $T1 = 1$ and $\|T^n\|_1 = 1 + \varepsilon$ for all $n \geqslant 1$, and a function f in L_1 such that the individual ergodic limit

$$\lim_n \frac{1}{n} \sum_{i=0}^{n-1} T^i f(x)$$

does not exist almost everywhere on a certain measurable subset of positive measure.

In this paper, however, we shall prove the following individual ergodic theorem.

THEOREM. *Let $(X, \mathfrak{F}, \mu)$ be a finite measure space and $L_p = L_p(X, \mathfrak{F}, \mu)$, $1 \leqslant p \leqslant \infty$, the usual Banach spaces. Let T be a bounded linear operator on L_1 and τ its linear modulus in the sense of Chacon and Krengel* [4]. *Assume the conditions*:

$$\sup_n \left\| \frac{1}{n} \sum_{i=0}^{n-1} \tau^i \right\|_1 < \infty, \tag{1}$$

$$\sup_n \left\| \frac{1}{n} \sum_{i=0}^{n-1} \tau^i \right\|_\infty < \infty. \tag{2}$$

Then, for any $f \in L_\infty$, the ergodic limit

$$\lim_n \frac{1}{n} \sum_{i=0}^{n-1} T^i f(x)$$

exists for almost all $x \in X$.

PROOF. Let T^* and τ^* denote the adjoint operators of T and τ, respectively. Since $|T^* f| \leqslant \tau^* |f|$ (cf. [1]) and $\int \tau^* |f| \, d\mu = \int (\tau 1)|f| \, d\mu \leqslant \|\tau\|_\infty \|f\|_1$ for all

Received by the editors November 17, 1976 and, in revised form, January 31, 1977.
AMS (MOS) subject classifications (1970). Primary 47A35; Secondary 28A65.
Key words and phrases. Individual and mean ergodic theorems, Chacon's general ratio ergodic theorem, finite measure space, linear operator, linear modulus of a linear operator, norm conditions.

$f \in L_\infty$, T^* and τ^* can be extended to bounded linear operators S and σ on L_1, respectively. It is easily seen that

$$S^* = T \quad (\text{on } L_\infty) \quad \text{and} \quad \sigma^* = \tau \quad (\text{on } L_\infty).$$

Hence it follows that the linear modulus of S is σ and that

$$\sup_n \left\| \frac{1}{n} \sum_{i=0}^{n-1} \sigma^i \right\|_1 = \sup_n \left\| \frac{1}{n} \sum_{i=0}^{n-1} \tau^i \right\|_\infty < \infty.$$

We now define two functions u and v in L_∞ by the relations:

$$u(x) = \limsup_n \frac{1}{n} \sum_{i=0}^{n-1} \tau^{*i} 1(x) \qquad (x \in X)$$

and

$$v(x) = \limsup_n \frac{1}{n} \sum_{i=0}^{n-1} \tau^i 1(x) \qquad (x \in X).$$

It may be readily seen that $\tau^* u \geqslant u$ and $\tau v \geqslant v$. Thus if we set

$$s(x) = \lim_n \frac{1}{n} \sum_{i=0}^{n-1} \tau^{*i} u(x) \qquad (x \in X)$$

and

$$t(x) = \lim_n \frac{1}{n} \sum_{i=0}^{n-1} \tau^i v(x) \qquad (x \in X),$$

then

(3) $$\tau^* s = s \quad \text{and} \quad \tau t = t.$$

Let $Y = \{x : s(x) > 0\}$ and $Z = X - Y$. Then, by [5] and [7], we have:

(i) $f \in L_1(Z)$ implies $\tau f \in L_1(Z)$,

(ii) if $f \in L_1(Z)$ then $\lim_n \|(1/n)\Sigma_{i=0}^{n-1} \tau^i f\|_1 = 0$,

(iii) $\lim_n \|\tau^n f\|_1 / n = 0$ for every $f \in L_1$.

We shall now divide the proof of the theorem into several steps, since it is rather long.

Step I. *If* $f \in L_\infty(Z)$ *then* $\lim_n (1/n)\Sigma_{i=0}^{n-1} T^i f(x) = 0$ *for almost all* $x \in X$.

To prove this, we may and do assume without loss of generality that $0 \leqslant f \leqslant 1$ on Z, and it is enough to show that the function h defined by

$$h(x) = \limsup_n \frac{1}{n} \sum_{i=0}^{n-1} \tau^i f(x) \qquad (x \in X)$$

satisfies $h(x) = 0$ almost everywhere on X. In fact, we notice that $0 \leqslant h \in L_\infty(Z)$ ($\subset L_1(Z)$), by (2) and (i), and that $\tau h \geqslant h$. Hence (ii) implies that

$$\|h\|_1 \leqslant \lim_n \left\| \frac{1}{n} \sum_{i=0}^{n-1} \tau^i h \right\|_1 = 0,$$

and therefore $h(x) = 0$ almost everywhere on X.

Step II. *For any* $f \in L_\infty$ *the limit* $\lim_n (1/n)\Sigma_{i=0}^{n-1} T^i f(x)$ *exists for almost all* $x \in Y$.

To see this, we first notice that

$$\int |Tf|s\,d\mu \leqslant \int (\tau|f|)s\,d\mu = \int |f|\tau^* s\,d\mu = \int |f|s\,d\mu$$

for all $f \in L_1$. Thus T may be regarded as a linear contraction operator on $L_1(Y, s\,d\mu)$, since $L_1 = L_1(X, \mathfrak{F}, \mu)$ is a dense subspace of $L_1(Y, s\,d\mu)$. Therefore we can apply Chacon's general ratio ergodic theorem [2], [3] to T to infer that, for any $f \in L_\infty = L_\infty(X, \mathfrak{F}, \mu)$ $(\subset L_1(Y, s\,d\mu))$, the limit

$$\lim_n \frac{1}{n} \sum_{i=0}^{n-1} T^i f(x) = \lim_n t(x) \sum_{i=0}^{n-1} T^i f(x) \Big/ \sum_{i=0}^{n-1} \tau^i t(x)$$

exists almost everywhere on $Y \cap \{x\colon t(x) > 0\}$. On the other hand it is immediate from the definition of t that, for any $f \in L_\infty$,

$$\lim_n \frac{1}{n} \sum_{i=0}^{n-1} T^i f(x) = 0$$

almost everywhere on $\{x : t(x) = 0\}$. Thus Step II is established.

Step III. *If* $f \in L_\infty$ *and* $\lim_n \|(1/n)\Sigma_{i=0}^{n-1} T^i f\|_1 = 0$, *then* $\lim_n (1/n)\Sigma_{i=0}^{n-1} T^i f(x) = 0$ *for almost all* $x \in X$.

To prove this, define

$$f_0(x) = \limsup_n \left| \frac{1}{n} \sum_{i=0}^{n-1} T^i f(x) \right| \qquad (x \in X). \tag{4}$$

Clearly, $0 \leqslant f_0 \in L_\infty$, and it follows from Step II that $f_0(x) = 0$ for almost all $x \in Y$. For each $k \geqslant 1$, put

$$f_k = \frac{1}{k} \sum_{i=0}^{k-1} T^i f. \tag{5}$$

By an easy computation we then have

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} T^i f - \frac{1}{n} \sum_{i=0}^{n-1} T^i f_k \right| = \frac{1}{n} \left| (T^n - I) \sum_{i=1}^{k-1} \left(1 - \frac{i}{k}\right) T^{i-1} f \right|.$$

To see that

$$\lim_n \left| \frac{1}{n} \sum_{i=0}^{n-1} T^i f(x) - \frac{1}{n} \sum_{i=0}^{n-1} T^i f_k(x) \right| = 0$$

for almost all $x \in X$, define

$$h(x) = \left| \sum_{i=1}^{k-1} \left(1 - \frac{i}{k}\right) T^{i-1} f(x) \right| \qquad (x \in X)$$

and

$$\tilde{h}(x) = \limsup_n \frac{1}{n} \tau^n h(x) \qquad (x \in X).$$

Then we have

$$\limsup_n \frac{1}{n}\left|T^n\left(\sum_{i=1}^{k-1}(1-i/k)\,T^{i-1}f\right)(x)\right| \leqslant \tilde{h}(x)$$

for almost all $x \in X$, and furthermore

$$\tau\tilde{h}(x) = \tau\left(\limsup_n \frac{1}{n}\,\tau^n h(x)\right) \geqslant \limsup_n \frac{1}{n}\,\tau^{n+1}h(x) = \tilde{h}(x)$$

for almost all $x \in X$. Here we can apply Step II to τ instead of T and obtain that $\tilde{h}(x) = 0$ for almost all $x \in Y$. Therefore $\tilde{h} \in L_\infty(Z)$ $(\subset L_1(Z))$, and so

$$\|\tilde{h}\|_1 \leqslant \lim_n \left\|\frac{1}{n}\sum_{i=0}^{n-1}\tau^i\tilde{h}\right\|_1 = 0,$$

by (ii). Consequently, $\tilde{h}(x) = 0$ for almost all $x \in X$.

It then follows that, for almost all $x \in X$,

$$\limsup_n \left|\frac{1}{n}\sum_{i=0}^{n-1}T^i f(x) - \frac{1}{n}\sum_{i=0}^{n-1}T^i f_k(x)\right| = 0 \qquad (k \geqslant 1).$$

Thus, for almost all $x \in X$,

$$\begin{aligned} f_0(x) = \limsup_n \left|\frac{1}{n}\sum_{i=0}^{n-1}T^i f_k(x)\right| &\leqslant \limsup_n \frac{1}{n}\sum_{i=0}^{n-1}\tau^i|f_k|(x) \\ &\leqslant \limsup_n \frac{1}{n}\sum_{i=0}^{n-1}\tau^i f_k^1(x) + \limsup_n \frac{1}{n}\sum_{i=0}^{n-1}\tau^i f_k^2(x), \end{aligned}$$

where $f_k^1 = |f_k|1_Z$ and $f_k^2 = |f_k|1_Y$. By this and the argument in Step I, we obtain that

$$f_0(x) \leqslant \limsup_n \frac{1}{n}\sum_{i=0}^{n-1}\tau^i f_k^2(x) \tag{6}$$

for almost all $x \in X$, because $f_k^1 \in L_\infty(Z)$.

Let us put, for each $k \geqslant 1$,

$$\tilde{f}_k(x) = \limsup_n \frac{1}{n}\sum_{i=0}^{n-1}\tau^i f_k^2(x) \qquad (x \in X).$$

It follows that $0 \leqslant \tilde{f}_k \in L_\infty$ and that $\tau\tilde{f}_k \geqslant \tilde{f}_k$. Hence, writing $g_k^1 = \tilde{f}_k 1_Z$ and $g_k^2 = \tilde{f}_k 1_Y$, and applying (ii) to $g_k^1 \in L_\infty(Z)$ $(\subset L_1(Z))$, we get

$$\begin{aligned} \|\tilde{f}_k\|_1 &\leqslant \lim_n \left\|\frac{1}{n}\sum_{i=0}^{n-1}\tau^i\tilde{f}_k\right\|_1 \\ &\leqslant \lim_n \left\|\frac{1}{n}\sum_{i=0}^{n-1}\tau^i g_k^1\right\|_1 + \lim_n \left\|\frac{1}{n}\sum_{i=0}^{n-1}\tau^i g_k^2\right\|_1 \\ &= \lim_n \left\|\frac{1}{n}\sum_{i=0}^{n-1}\tau^i g_k^2\right\|_1 \leqslant \left(\sup_n \left\|\frac{1}{n}\sum_{i=0}^{n-1}\tau^i\right\|_1\right)\|g_k^2\|_1. \end{aligned}$$

On the other hand, applying Step II to τ instead of T, we observe that

$$g_k^2(x) = \lim_n \left(\frac{1}{n} \sum_{i=0}^{n-1} \tau^i f_k^2(x) \right) 1_Y(x)$$

for almost all $x \in X$. Hence, by Fatou's lemma,

$$\| g_k^2 \|_1 \leqslant \liminf_n \left\| \left(\frac{1}{n} \sum_{i=0}^{n-1} \tau^i f_k^2 \right) 1_Y \right\|_1$$

$$\leqslant \left\{ \sup_n \left\| \frac{1}{n} \sum_{i=0}^{n-1} \tau^i \right\|_1 \right\} \| f_k^2 \|_1.$$

By this and the fact that

$$\lim_k \| f_k^2 \|_1 = \lim_k \left\| \left(\frac{1}{k} \sum_{i=0}^{k-1} T^i f \right) 1_Y \right\|_1 = 0,$$

we obtain that $\lim_k \| g_k^2 \|_1 = 0$, and hence that $\lim_k \| \tilde{f}_k \|_1 = 0$. Since $0 \leqslant f_0(x) \leqslant \tilde{f}_k(x)$ for almost all $x \in X$, it must follow that $\| f_0 \|_1 = 0$. So $f_0(x) = 0$ almost everywhere on X, and this establishes Step III.

Step IV. *For any $f \in L_\infty$ there exists a function $g \in L_\infty$ such that $Tg = g$ and* $\lim_n \| (1/n) \Sigma_{i=0}^{n-1} T^i (f - g) \|_1 = 0$.

To prove this, let $f \in L_\infty$ be given. Then, by (2) and Theorem IV.8.9 in [6], the set $\{(1/n)\Sigma_{i=0}^{n-1} T^i f \colon n \geqslant 1\}$ is weakly sequentially compact in L_1, and by (iii), $\lim_n \| T^n f \|_1 / n \leqslant \lim_n \| \tau^n |f| \|_1 / n = 0$. Hence, a well-known mean ergodic theorem (cf. Theorem VIII.5.1 in [6]) shows that there exists a function $g \in L_1$ such that $Tg = g$ and

$$\lim_n \left\| \frac{1}{n} \sum_{i=0}^{n-1} T^i f - g \right\|_1 = 0.$$

Clearly $g \in L_\infty$, by condition (2), and hence Step IV is established.

Step V. *We shall now conclude the proof of the theorem as follows.*

Let $f \in L_\infty$ be given, and using Step IV, write $f = g + h$ where $Tg = g \in L_\infty$ and h $(\in L_\infty)$ satisfies $\lim_n \| (1/n) \Sigma_{i=0}^{n-1} T^i h \|_1 = 0$. Then, by Step III, we observe that

$$\lim_n \frac{1}{n} \sum_{i=0}^{n-1} T^i f(x) = g(x)$$

for almost all $x \in X$. Hence the proof is completed.

REMARK. In [8], an analogous result is proved for a strongly continuous one-parameter semigroup $\{T_t\}_{0<t<\infty}$ of positive linear operators on L_1 of a finite measure space.

EXAMPLE. We shall construct a positive linear operator T on L_1 of a finite measure space which satisfies the following norm conditions:

$$\sup_n \left\| \frac{1}{n} \sum_{i=0}^{n-1} T^i \right\|_1 < \infty \quad \text{and} \quad \sup_n \|T^n\|_1 = \infty, \tag{7}$$

$$\sup_n \left\| \frac{1}{n} \sum_{i=0}^{n-1} T^i \right\|_\infty < \infty \quad \text{and} \quad \sup_n \|T^n\|_\infty = \infty. \tag{8}$$

Put $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, $a_n = 4a_{n-1}$ $(n \geqslant 3)$, and $b_n = \Sigma_{i=0}^n a_i$ $(n \geqslant 0)$. Write $X = \{(n, i): n \geqslant 1 \text{ and } 1 \leqslant i \leqslant b_n\}$, and let $\mathfrak{F}$ be all possible subsets of X and λ the measure on $(X, \mathfrak{F})$ defined by $\lambda(\{(1, 1)\}) = 1$ and, for $n \geqslant 2$,

$$\lambda(\{(n, i)\}) = \begin{cases} 1/2^{n-1}, & \text{if } 1 \leqslant i \leqslant a_n, \\ \lambda(\{(n-1, i-a_n)\}), & \text{if } a_n < i \leqslant b_n. \end{cases}$$

Put $\mu(\{(n, i)\}) = (2/4^n)\lambda(\{(n, i)\})$. Then it follows from an easy computation that $(X, \mathfrak{F}, \mu)$ is a probability measure space. Define a point mapping φ from X to X by the relation:

$$\varphi((n, i)) = \begin{cases} (n, i+1), & \text{if } 1 \leqslant i < b_n, \\ (n+1, 1), & \text{if } i = b_n. \end{cases}$$

Then $X = \{\varphi^n((1, 1)) : n \geqslant\}$, and thus if we set, for convenience's sake, $\varphi^n((1, 1)) = n$, then $X = \{n : n \geqslant 0\}$. Define a positive linear operator S on $L_1(X, \mathfrak{F}, \mu)$ by the relation:

$$Sf(n) = f(n+1) \qquad (f \in L_1(X, \mathfrak{F}, \mu), n \in X).$$

Then we have $\sup_n \|(1/n)\Sigma_{i=0}^{n-1} S^i\|_1 \leqslant 4$, since

$$\frac{\mu(\{(n, i) : 1 \leqslant i \leqslant b_n\})}{b_n \mu(\{(n, 1)\})} < 2 \qquad (n \geqslant 1).$$

It is clear that $\sup_n \|S^n\|_1 = \infty$.

Next, let h be the function in $L_\infty(X, \mathfrak{F}, \mu)$ defined by $h(n) = 2^{-i+1}$, where $b_{i-1} < n + 1 \leqslant b_i$. Since

$$\left(\sum_{i=0}^{b_n} h(i) \right) / (b_n + 1) h(b_n) < 4 \qquad (n \geqslant 0),$$

it follows that

$$\sup_n \left\| \left(\frac{1}{n} \sum_{i=0}^{n-1} S^i h \right) / h \right\|_\infty \leqslant 4,$$

and also that $\sup_n \|(S^n h)/h\|_\infty = \infty$. Hence, if we define a positive linear operator T on $L_1(X, \mathfrak{F}, h\, d\mu)$ by the relation:

$$Tf = S(fh)/h \qquad (f \in L_1(X, \mathfrak{F}, h\, d\mu)),$$

then it is easily seen that T satisfies norm conditions (7) and (8).

ACKNOWLEDGEMENT. In conclusion, the author would like to express his gratitude to the referee for helpful comments which improved the paper.

BIBLIOGRAPHY

1. M. A. Akcoglu and A. Brunel, *Contractions on L_1-spaces*, Trans. Amer. Math. Soc. 155 (1971), 315–325.

2. R. V. Chacon, *Operator averages*, Bull. Amer. Math. Soc. 68 (1962), 351–353.

3. _____, *Convergence of operator averages*, Ergodic Theory (Proc. Internat. Sympos., Tulane Univ., New Orleans, La., 1961), Academic Press, New York, 1963, pp. 89–120.

4. R. V. Chacon and U. Krengel, *Linear modulus of a linear operator*, Proc. Amer. Math. Soc. 15 (1964), 553–559.

5. Y. Derriennic and M. Lin, *On invariant measures and ergodic theorems for positive operators*, J. Functional Anal. 13 (1973), 252–267.

6. N. Dunford and J. T. Schwartz, *Linear operators*. I: *General theory*, Pure and Appl. Math., vol. 7, Interscience, New York, 1958.

7. R. Sato, *Ergodic properties of bounded L_1-operators*, Proc. Amer. Math. Soc. 39 (1973), 540–546.

8. _____, *Ergodic theorems for semigroups of positive operators*, J. Math. Soc. Japan (to appear).

DEPARTMENT OF MATHEMATICS, JOSAI UNIVERSITY, SAKADO, SAITAMA, 350-02 JAPAN