A STABILITY THEOREM ON QUASI-REFLEXIVE OPERATORS

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Abstract. A range-closed bounded linear operator between Banach spaces is quasi-reflexive if both its kernel and cokernel are quasi-reflexive spaces. Under suitable conditions, if an operator is sufficiently close to a quasi-reflexive operator, it is itself quasi-reflexive.

1. Introduction. Let X and Y be Banach spaces. In [1], Civin and Yood defined that X is quasi-reflexive of order n if X**/J(X) is of finite dimension n, where X** is the second conjugate space of X and J: X → X** is the natural injection. A quasi-reflexive space of order 0 is simply a reflexive space. Let B(X, Y) be the Banach space of bounded linear operators from X to Y. An operator T ∈ B(X, Y) is called quasi-reflexive of type (m, n) if it has closed range and if Ker(T) and Coker(T) are quasi-reflexive of order m and n, respectively. In [3], such operators are called generalized Fredholm operators when the kernels and cokernels are both reflexive spaces, and a rather extensive theory for such operators is developed. We shall prove that under suitable conditions, if an operator is sufficiently close to a quasi-reflexive operator, it is itself a quasi-reflexive operator; and if φ(T) = m − n for a quasi-reflexive T of type (m, n), and S is close to T, then φ(S) = φ(T).

2. An auxiliary lemma. Recall that a closed subspace of X splits if it has a closed complementary subspace in X. We consider the following two conditions on Banach spaces:

(C) Every quasi-reflexive subspace splits.

(C") Every closed subspace with quasi-reflexive quotient space splits.

Lemma. Let X be a Banach space satisfying condition (C") Let M and N be closed subspaces of X, and M + N their vector sum. Assume that M + N is closed and that X/M is quasi-reflexive. Then M + N splits in X.

Proof. It is assumed that M + N is a closed subspace of X. By (C"), we need only show that X/(M + N) is quasi-reflexive. For a Banach space Y, let Y = Y**/J(Y). There is a natural onto map p: X/M → X/(M + N) which induces an onto map p: X/M → X/(M + N). As X/M is finite dimensional, X/(M + N) is finite dimensional. Q.E.D.

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3. The Theorem. It is easy to see that if $X$ is quasi-reflexive of order $n$, and $Y$ is isomorphic to $X$, then $Y$ is quasi-reflexive of order $n$. If $X$ is a closed subspace of $Y$, then $Y$ is quasi-reflexive of order $n$ if and only if $X$ and $Y/X$ are quasi-reflexive of order $m$ and $k$, respectively, and $n = m + k$ [1, Corollary 4.2]. Recall that a bounded linear operator is called double-splitting if it has closed range and both its kernel and cokernel split.

**Theorem.** Let $T \in B(X, Y)$ be double-splitting and quasi-reflexive of type $(m, n)$. Assume that $X$ and $Y$ satisfy condition $(C’)$. Let $S \in B(X, Y)$ be range-closed and sufficiently close to $T$. Then $S$ is double-splitting and quasi-reflexive of type $(m', n')$ with $m' - n' = m - n$.

**Proof.** Write $X = \text{Ker}(T) \oplus M$ and $Y = T(X) \oplus N$, where $M$ and $N \cong \text{Coker}(T)$ are closed complementary subspaces of $\text{Ker}(T)$ and $T(X)$, respectively. Following [2, Theorem 4, p. 122], let $p: M \oplus N \to M$ and $q: M \oplus N \to N$ be the projections. Consider the map $f(S) = S \cdot p + q: M \oplus N \to Y$, for $S \in B(X, Y)$. Then $f$ is a continuous map from $B(X, Y)$ into $B(M \oplus N, Y)$. By the open mapping theorem, $f(T)$ is an isomorphism, hence $f(S)$ is an isomorphism and $Y = f(S)(M) \oplus f(S)(N)$ for $S$ close to $T$. But $f(S)(M) = S(M)$ and $f(S)(N) = N \cong \text{Coker}(T)$, so that $Y \cong S(M) \oplus \text{Coker}(T)$.

Since $S$ is one-to-one on $M$, $\text{Ker}(S) \oplus M = \text{Ker}(S) + M$, and since $X/M = \text{Ker}(T)$ is quasi-reflexive, $\text{Ker}(S) \oplus M$ has a closed complementary subspace $V$ in $X$ by the Lemma, and then

$$X = \text{Ker}(S) \oplus M \oplus V = \text{Ker}(T) \oplus M.$$  

Thus $\text{Ker}(S)$ splits in $X$ and $\text{Ker}(S) \oplus V = \text{Ker}(T)$. As $\text{Ker}(T)$ is quasi-reflexive, $\text{Ker}(S)$ and $V$ are quasi-reflexive. By assumption, $S(X)$ is closed, and since $S: M \oplus V \to S(X)$ is bijective and continuous, $S$ is an isomorphism from $M \oplus V$ onto $S(X)$. Therefore $S(V)$ is closed and isomorphic to $V$ so that $S(V)$ is quasi-reflexive. It is clear that $S(M)$ is closed and has a quasi-reflexive complementary subspace in $Y$, and the image $S(X) = S(M) + S(V)$ of $S$ splits in $Y$ by the Lemma. We have proved that $S$ is double-splitting. Thus

$$Y = S(M) \oplus S(V) \oplus \text{Coker}(S) \cong S(M) \oplus \text{Coker}(T),$$

and $\text{Coker}(T) = \text{Coker}(S) \oplus S(V)$. It follows that $\text{Coker}(S)$ is quasi-reflexive and therefore $S$ is quasi-reflexive. Since $\text{Ker}(T) \cong \text{Ker}(S) \oplus V$, $\text{Coker}(T) \cong \text{Coker}(S) \oplus S(V)$ and $V \cong S(V)$, we have $m' - n' = m - n$. Q.E.D.

We notice that if $X$ satisfies condition $(C’)$, and $Y$ satisfies condition $(C’’)$, then any quasi-reflexive operator is double-splitting.

**Corollary.** Let $X$ be a Banach space satisfying $(C’)$ and $(C’’)$, and $Y$ a Banach space satisfying $(C’’$). Let $T \in B(X, Y)$ be quasi-reflexive of type $(m, n)$. If $S \in B(X, Y)$ is range-closed and close to $T$, then $S$ is quasi-reflexive of type $(m', n')$ with $m' - n' = m - n$. 

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4. Remarks. If we replace ‘quasi-reflexive’ in conditions (C') and (C'') by ‘reflexive’, we have a corresponding theorem for generalized Fredholm operators as defined in [3]. If we consider only finite dimensional subspaces, condition (C') and (C'') are void, the Theorem assumes a much stronger form and gives the result for classical Fredholm operators [2, Theorem 4, p. 122].

Let $\phi(T) = m - n$ for a quasi-reflexive operator $T$ of type $(m, n)$. If $T \in B(X, Y)$ and $S \in B(Y, Z)$ are quasi-reflexive operators and $ST \in B(X, Z)$ is range-closed, then $ST$ is also quasi-reflexive and $\phi(ST) = \phi(S) + \phi(T)$. In fact, if one employs the functor $- \cdot B \rightarrow B$ in [3], one can show that a range-closed operator $T \in B(X, Y)$ is quasi-reflexive if and only if $\overline{T} \in B(X, \overline{Y})$ is Fredholm, and then $\phi(T) = \text{ind}(\overline{T})$, the index of $\overline{T}$.

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