PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 65, Number 2, August 1977

A STABILITY THEOREM ON QUASI-REFLEXIVE OPERATORS

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ABSTRACT. A range-closed bounded linear operator between Banach spaces is quasi-reflexive if both its kernel and cokernel are quasi-reflexive spaces. Under suitable conditions, if an operator is sufficiently close to a quasi-reflexive operator, it is itself quasi-reflexive.

1. **Introduction**. Let X and Y be Banach spaces. In [1], Civin and Yood defined that X is *quasi-reflexive of order n* if $X^{**}/J(X)$ is of finite dimension n, where X^{**} is the second conjugate space of X and J: $X \to X^{**}$ is the *natural injection*. A quasi-reflexive space of order 0 is simply a reflexive space. Let $B(X, Y)$ be the Banach space of bounded linear operators from X to Y. An operator $T \in B(X, Y)$ is called *quasi-reflexive of type* (m, n) if it has closed range and if $\operatorname{Ker}(T)$ and $\operatorname{Coker}(T)$ are quasi-reflexive of order m and n, respectively. In [3], such operators are called generalized Fredholm operators when the kernels and cokernels are both reflexive spaces, and a rather extensive theory for such operators is developed. We shall prove that under suitable conditions, if an operator is sufficiently close to a quasi-reflexive operator, it is itself a quasi-reflexive operator; and if $\phi(T) = m - n$ for a quasi-reflexive T of type (m, n), and S is close to T, then $\phi(S) = \phi(T)$.

2. **An auxilliary lemma.** Recall that a closed subspace of X *splits* if it has a closed complementary subspace in X. We consider the following two conditions on Banach spaces:

(C′) *Every quasi-reflexive subspace splits.*

(C″) *Every closed subspace with quasi-reflexive quotient space splits.*

LEMMA. *Let X be a Banach space satisfying condition* (C″). *Let M and N be closed subspaces of X, and $M + N$ their vector sum. Assume that $M + N$ be closed and that X/M is quasi-reflexive. Then $M + N$ splits in X.*

PROOF. It is assumed that $M + N$ is a closed subspace of X. By (C″), we need only show that $X/(M + N)$ is quasi-reflexive. For a Banach space Y, let $\bar{Y} = Y^{**}/J(Y)$. There is a natural onto map p: $X/M \to X/(M + N)$ which induces an onto map $\bar{p}$: $\overline{X/M} \to \overline{X/(M + N)}$. As $\overline{X/M}$ is finite dimensional, $\overline{X/(M + N)}$ is finite dimensional. Q.E.D.

Received by the editors September 28, 1976.
AMS (MOS) subject classifications (1970). Primary 47B05.
Key words and phrases. Banach spaces, complementary subspace, quasi-reflexive, double-splitting, kernel, cokernel.

3. **The Theorem.** It is easy to see that if X is quasi-reflexive of order n, and Y is isomorphic to X, then Y is quasi-reflexive of order n. If X is a closed subspace of Y, then Y is quasi-reflexive of order n if and only if X and Y/X are quasi-reflexive of order m and k, respectively, and $n = m + k$ [1, Corollary 4.2]. Recall that a bounded linear operator is called *double-splitting* if it has closed range and both its kernel and cokernel split.

THEOREM. *Let $T \in B(X, Y)$ be double-splitting and quasi-reflexive of type (m, n). Assume that X and Y satisfy condition* (C″). *Let $S \in B(X, Y)$ be range-closed and sufficiently close to T. Then S is double-splitting and quasi-reflexive of type (m', n') with $m' - n' = m - n$.*

PROOF. Write $X = \mathrm{Ker}(T) \oplus M$ and $Y = T(X) \oplus N$, where M and $N \cong \mathrm{Coker}(T)$ are closed complementary subspaces of $\mathrm{Ker}(T)$ and $T(X)$, respectively. Following [2, Theorem 4, p. 122], let p: $M \oplus N \to M$ and q: $M \oplus N \to N$ be the projections. Consider the map $f(S) = S \cdot p + q$: $M \oplus N \to Y$, for $S \in B(X, Y)$. Then f is a continuous map from $B(X, Y)$ into $B(M \oplus N, Y)$. By the open mapping theorem, $f(T)$ is an isomorphism, hence $f(S)$ is an isomorphism and $Y = f(S)(M) \oplus f(S)(N)$ for S close to T. But $f(S)(M) = S(M)$ and $f(S)(N) = N \cong \mathrm{Coker}(T)$, so that $Y \cong S(M) \oplus \mathrm{Coker}(T)$.

Since S is one-to-one on M, $\mathrm{Ker}(S) \oplus M = \mathrm{Ker}(S) + M$, and since $X/M = \mathrm{Ker}(T)$ is quasi-reflexive, $\mathrm{Ker}(S) \oplus M$ has a closed complementary subspace V in X by the Lemma, and then

$$X = \mathrm{Ker}(S) \oplus M \oplus V = \mathrm{Ker}(T) \oplus M.$$

Thus $\mathrm{Ker}(S)$ splits in X and $\mathrm{Ker}(S) \oplus V = \mathrm{Ker}(T)$. As $\mathrm{Ker}(T)$ is quasi-reflexive, $\mathrm{Ker}(S)$ and V are quasi-reflexive. By assumption, $S(X)$ is closed, and since S: $M \oplus V \to S(X)$ is bijective and continuous, S is an isomorphism from $M \oplus V$ onto $S(X)$. Therefore $S(V)$ is closed and isomorphic to V so that $S(V)$ is quasi-reflexive. It is clear that $S(M)$ is closed and has a quasi-reflexive complementary subspace in Y, and the image $S(X) = S(M) + S(V)$ of S splits in Y by the Lemma. We have proved that S is double-splitting. Thus

$$Y \cong S(M) \oplus S(V) \oplus \mathrm{Coker}(S) \cong S(M) \oplus \mathrm{Coker}(T),$$

and $\mathrm{Coker}(T) \cong \mathrm{Coker}(S) \oplus S(V)$. It follows that $\mathrm{Coker}(S)$ is quasi-reflexive and therefore S is quasi-reflexive. Since $\mathrm{Ker}(T) \cong \mathrm{Ker}(S) \oplus V$, $\mathrm{Coker}(T) \cong \mathrm{Coker}(S) \oplus S(V)$ and $V \cong S(V)$, we have $m' - n' = m - n$. Q.E.D.

We notice that if X satisfies condition (C′), and Y satisfies condition (C″), then any quasi-reflexive operator is double-splitting.

COROLLARY. *Let X be a Banach space satisfying* (C′) *and* (C″), *and Y a Banach space satisfying* (C″). *Let $T \in B(X, Y)$ be quasi-reflexive of type (m, n). If $S \in B(X, Y)$ is range-closed and close to T, then S is quasi-reflexive of type (m', n') with $m' - n' = m - n$.*

4. **Remarks.** If we replace '*quasi-reflexive*' in conditions (C′) and (C″) by '*reflexive*', we have a corresponding theorem for generalized Fredholm operators as defined in [3]. If we consider only *finite dimensional* subspaces, condition (C′) and (C″) are void, the Theorem assumes a much stronger form and gives the result for classical Fredholm operators [2, Theorem 4, p. 122].

Let $\phi(T) = m - n$ for a quasi-reflexive operator T of type (m, n). If $T \in B(X, Y)$ and $S \in B(Y, Z)$ are quasi-reflexive operators and $ST \in B(X, Z)$ is range-closed, then ST is also quasi-reflexive and $\phi(ST) = \phi(S) + \phi(T)$. In fact, if one employs the functor $\bar{-}\colon B \to B$ in [3], one can show that a range-closed operator $T \in B(X, Y)$ is quasi-reflexive if and only if $\bar{T} \in B(\bar{X}, \bar{Y})$ is Fredholm, and then $\phi(T) = \operatorname{ind}(\bar{T})$, the index of $\bar{T}$.

References

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