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AN INEQUALITY FOR FUNCTIONS OF EXPONENTIAL TYPE NOT VANISHING IN A HALF-PLANE

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ABSTRACT. Let $f(z)$ be an entire function of order 1, type τ having no zero in $\operatorname{Im} z < 0$. If $h_f(-\pi/2) = \tau$, $h_f(\pi/2) \leq 0$ then it is known that $\sup_{-\infty<x<\infty}|f'(x)| \geq (\tau/2)\sup_{-\infty<x<\infty}|f(x)|$. In this paper we consider the case when $f(z)$ has no zero in $\operatorname{Im} z < k$, $k \leq 0$ and obtain a sharp result.

1. If $f(z)$ is an entire function of exponential type τ and $|f(x)| \leq M$ for real x, then according to a well-known theorem due to S. N. Bernstein [1, p. 206]

$$|f'(x)| \leq M\tau, \qquad -\infty < x < \infty. \tag{1.1}$$

If $h_f(\pi/2) = 0$,

$$h_f(\theta) = \limsup_{r\to\infty} \frac{\log|f(re^{i\theta})|}{r}$$

is the indicator function of $f(z)$, and $f(x + iy) \neq 0$ for $y > 0$, then it has been proved by Boas [2] that (1.1) can be replaced by

$$|f'(x)| \leq M\tau/2, \qquad -\infty < x < \infty. \tag{1.2}$$

This result of Boas is in fact a generalization of the Erdös conjecture proved by Lax [4] because the class of asymmetric entire functions of exponential type τ includes all functions $p(e^{iz})$ where $p(z)$ is a polynomial of degree $n \leq [\tau]$ and $p(z) \neq 0$ in $|z| < 1$.

For polynomials having all their zeros in $|z| \leq 1$, we have the following result due to Turán [6].

THEOREM A. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then*

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.3}$$

As a generalization of Theorem A, an inequality analogous to (1.3) for entire functions of order 1, and type τ has been obtained by Rahman [5] and for polynomials having all its zeros in $|z| \leq K$, $K \geq 0$ by Govil [3]. In this

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paper we generalize the result due to Rahman [5] and due to Govil [3] (in the case $K \geqslant 1$) and prove the following.

THEOREM. *Let $f(z)$ be an entire function of order* 1, *type τ having all its zeros in* $\operatorname{Im} z \geqslant k$, $k \leqslant 0$. *If $h_f(\pi/2) \leqslant 0$, $h_f(-\pi/2) = \tau$ then*

$$\sup_{-\infty<x<\infty} |f'(x)| \geqslant \frac{\tau}{1+\exp(\tau|k|)} \sup_{-\infty<x<\infty} |f(x)|.$$

The result is best possible with equality for the function

$$f(z) = \left(\frac{e^{i\tau z} - e^{-\tau k}}{1+e^{-\tau k}}\right).$$

2. For the proof of the theorem, we need the following lemmas.

LEMMA 1. *If $f(z)$ is an entire function of exponential type τ and $|f(x)| \leqslant M$, $-\infty < x < \infty$, then*

$$|f(x+iy)| \leqslant Me^{\tau|y|}, \qquad -\infty < x < \infty, -\infty < y < \infty. \tag{2.1}$$

Lemma 1 is a simple consequence of the Phragmén-Lindelöf principle and follows immediately from a result due to Pólya and Szegö (see [1, p. 82, Theorem 6.2.4]).

LEMMA 2. *Let $f(z)$ be an entire function of order* 1, *type* τ, $h_f(\pi/2) \leqslant 0$, $|f(x)| \leqslant M$, $-\infty < x < \infty$, *and let* $g(z) = e^{i\tau z}\operatorname{con}\{f(\bar{z})\}$, *where* $\operatorname{con}\{f(\bar{z})\}$ *denotes the conjugate of $f(\bar{z})$. Then type $g \leqslant \tau$.*

PROOF OF LEMMA 2. If $g(z) = e^{i\tau z}\operatorname{con}\{f(\bar{z})\}$ is an entire function of order less than 1, then obviously type $g \leqslant \tau$, hence it is sufficient to prove the result when $g(z)$ is of order 1.

If $z = re^{i\theta}$ is a point of the upper half-plane, then

$$|g(re^{i\theta})| = e^{-\tau r\sin\theta}|f(re^{-i\theta})|,$$

which gives by Lemma 1,

$$\begin{aligned} |g(re^{i\theta})| &\leqslant e^{-\tau r\sin\theta}e^{\tau r\sin\theta} \sup_{-\infty<x<\infty} |f(x)| \\ &= \sup_{-\infty<x<\infty} |f(x)|. \end{aligned} \tag{2.2}$$

If $z = re^{i\theta}$ lies in the lower half-plane, the point $z = re^{-i\theta}$ will lie in the upper half-plane and since $h_f(\pi/2) \leqslant 0$, hence it follows by a result due to Pólya and Szegö (see [1, p. 82, Theorem 6.2.4]) that

$$\begin{aligned} |g(re^{i\theta})| &= e^{-\tau r\sin\theta}|f(re^{-i\theta})| \\ &\leqslant e^{-\tau r\sin\theta} \sup_{-\infty<x<\infty} |f(x)| \\ &\leqslant e^{\tau r} \sup_{-\infty<x<\infty} |f(x)|. \end{aligned} \tag{2.3}$$

On combining (2.2) and (2.3), we get

$$|g(re^{i\theta})| \leqslant e^{\tau r} \sup_{-\infty < x < \infty} |f(x)|, \qquad 0 \leqslant \theta < 2\pi,$$

which gives that type $g \leqslant \tau =$ type f, and Lemma 2 follows.

LEMMA 3. *If $f(z)$ is an entire function of order* 1, *type τ such that $h_f(-\pi/2) = \tau$, $h_f(\pi/2) \leqslant 0$, $f(z)$ has all its zeros in* $\operatorname{Im} z \geqslant k$, $k \leqslant 0$, *then*

$$\sup_{-\infty < x < \infty} |g'(x)| \leqslant e^{\tau|k|} \sup_{-\infty < x < \infty} |f'(x)|, \tag{2.4}$$

where as in Lemma 2, *$g(z)$ stands for $e^{i\tau z}\mathrm{con}\{f(\bar{z})\}$ and $\mathrm{con}\{f(\bar{z})\}$ for the conjugate of $f(\bar{z})$.*

PROOF OF LEMMA 3. Let $F(z) = f(z + ik)$ and $G(z) = e^{i\tau z}\mathrm{con}\{F(\bar{z})\} = e^{-\tau k}g(z - ik)$, where $\mathrm{con}\{F(\bar{z})\}$ denotes the conjugate of $\{F(\bar{z})\}$. Since $f(z)$ has all its zeros in $\operatorname{Im} z \geqslant k$, $k \leqslant 0$, $h_f(-\pi/2) = \tau$, $h_f(\pi/2) \leqslant 0$, the function $F(z)$ is an entire function of order 1, type τ, has no zero in $\operatorname{Im} z < 0$, $h_F(-\pi/2) = \tau$ and $h_F(\pi/2) \leqslant 0$. Therefore the function $F(z)$ belongs to the class P. Further $F_1(z) = e^{i\tau z/2}\mathrm{con}\{F(\bar{z})\}$ is an entire function of exponential type having no zero in $\operatorname{Im} z > 0$ and satisfying $h_{F_1}(\pi/2) \geqslant h_{F_1}(-\pi/2)$. Hence applying a result due to Levin (see [1, p. 129, Theorem 7.8.1]) to the function $F_1(z)$, we get

$$\left|e^{i\tau z/2}\mathrm{con}\{F(\bar{z})\}\right| \geqslant \left|e^{i\tau \bar{z}/2}\mathrm{con}\{F(z)\}\right| \quad \text{for } \operatorname{Im} z \geqslant 0,$$

which implies

$$\left|e^{i\tau \bar{z}/2}\mathrm{con}\{F(z)\}\right| \geqslant \left|e^{i\tau z/2}\mathrm{con}\{F(\bar{z})\}\right| \quad \text{for } \operatorname{Im} z \leqslant 0,$$

and which implies

$$\left|F(z)e^{-i\tau z/2}\right| \geqslant \left|e^{i\tau z/2}\mathrm{con}\{F(\bar{z})\}\right| \quad \text{for } \operatorname{Im} z \leqslant 0.$$

Thus

$$|F(z)| \geqslant \left|e^{i\tau z}\mathrm{con}\{F(\bar{z})\}\right| \quad \text{for } \operatorname{Im} z \leqslant 0.$$

Since $G(z) = e^{i\tau z}\mathrm{con}\{F(\bar{z})\}$, we get

$$|F(z)| \geqslant |G(z)| \quad \text{for } \operatorname{Im} z \leqslant 0. \tag{2.5}$$

For $k \leqslant 0$, let $F_k(z)$ denote the function $F(z + ik)$ and $G_k(z)$ the function $G(z + ik)$. Then the function $F_k(z)$ is an entire function of order 1, type τ. Also by Lemma 2, the function $G_k(z)$ is an entire function of exponential type $\leqslant \tau$. Since $F(z)$ has no zero in $\operatorname{Im} z < 0$, therefore $F_k(z)$ has no zero in $\operatorname{Im} z < -k$, and hence no zero in $\operatorname{Im} z < 0$, because $k \leqslant 0$. Further because $h_{F_k}(-\pi/2) = h_F(-\pi/2) = \tau$, $h_{F_k}(\pi/2) = h_F(\pi/2) \leqslant 0$, we get $h_{F_k}(-\pi/2) \geqslant h_{F_k}(\pi/2)$ and therefore $F_k(z)$ belongs to the class P. Thus $G_k(z)$ is an entire function of exponential type $\leqslant \tau$ and $F_k(z)$ an entire function of class P, order 1 and type τ. Also by (2.5) we have $|G_k(x)| \leqslant |F_k(x)|$, $-\infty < x < \infty$, hence applying a result due to Levin (see [1, p. 226, Theorem 11.7.2]) and the fact that differentiation is a B-operator, we get $|G_k'(x)| \leqslant |F_k'(x)|$, $-\infty < x < \infty$, which implies

$$(2.6)\qquad |G'(x+ik)| \leqslant |F'(x+ik)|, \qquad -\infty < x < \infty, k \leqslant 0.$$

Since $|F'(x+ik)| = |f'(x+2ik)|$, and $|G'(x+ik)| = e^{-\tau k}|g'(x)|$, (2.6) gives,

$$(2.7)\qquad |g'(x)| \leqslant e^{\tau k}|f'(x+2ik)|, \qquad -\infty < x < \infty.$$

Lastly applying the inequality (2.1) to $|f'(x+2ik)|$ and combining it with (2.7) we get

$$|g'(x)| \leqslant e^{\tau|k|} \sup_{-\infty<x<\infty} |f'(x)|,$$

from which the lemma follows.

LEMMA 4. *If $f(z)$ is an entire function of exponential type τ, then*

$$\sup_{-\infty<x<\infty} |f'(x)| + \sup_{-\infty<x<\infty} |g'(x)| \geqslant \tau \sup_{-\infty<x<\infty} |f(x)|,$$

where $g(z)$ is the same as defined in Lemma 3.

PROOF OF LEMMA 4. From the definition of $g(z)$ it follows that on real axes.

$$|g'(x)| = \left|e^{i\tau x}\,\overline{f'(x)} + i\tau e^{i\tau x}\,\overline{f(x)}\right|$$
$$\geqslant \tau|f(x)| - |f'(x)|.$$

Thus for $-\infty < x < \infty$, $|f'(x)| + |g'(x)| \geqslant \tau|f(x)|$, which gives

$$\sup_{-\infty<x<\infty} |f'(x)| + \sup_{-\infty<x<\infty} |g'(x)| \geqslant \tau \sup_{-\infty<x<\infty} |f(x)|,$$

and Lemma 4 is proved.

3. **Proof of the Theorem.** We have by Lemma 3,

$$(3.1)\qquad \sup_{-\infty<x<\infty} |g'(x)| \leqslant e^{\tau|k|} \sup_{-\infty<x<\infty} |f'(x)|.$$

Also by Lemma 4,

$$(3.2)\qquad \sup_{-\infty<x<\infty} |f'(x)| + \sup_{-\infty<x<\infty} |g'(x)| \geqslant \tau \sup_{-\infty<x<\infty} |f(x)|.$$

Combining (3.1) and (3.2) we get

$$\sup_{-\infty<x<\infty} |f'(x)| + e^{\tau|k|} \sup_{-\infty<x<\infty} |f'(x)| \geqslant \tau \sup_{-\infty<x<\infty} |f(x)|,$$

which implies

$$\sup_{-\infty<x<\infty} |f'(x)| \geqslant \frac{\tau}{1+e^{\tau|k|}} \sup_{-\infty<x<\infty} |f(x)|.$$

This completes the proof of the theorem.

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