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LINEAR ISOTOPIES IN E^3

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ABSTRACT. In this paper it is shown that if f and g are two PL embeddings of a finite complex K into E^3 so that there is an orientation-preserving homeomorphism h of E^3 with $h \circ f = g$, then there is a triangulation T of K and a linear isotopy h_t: $(K, T) \to E^3$ ($t \in [0, 1]$) so that $h_0 = f$ and $h_1 = g$.

1. **Introduction**. Let C be a finite complex with triangulation T and f and g be two linear embeddings of (C, T) into E^3 so that there is an orientation-preserving homeomorphism h of E^3 with $h \circ f = g$. The example in [4] shows that, unlike the situation in E^2 [3], one cannot hope to conclude that there is a continuous family of linear embeddings of (C, T) into E^3 starting at f and ending at g. It is known, however, that there is a PL isotopy of E^3, H_t ($t \in [0, 1]$), such that $H_0 = \mathrm{id}$ and $H_1 \circ f = g$. The existence of this PL isotopy shows that there is a continuous family of PL embeddings of C into E^3 starting at f and ending at g.

In this paper we show that under the conditions given in the first sentence of the introduction, there is some one triangulation T' of C and a continuous family of linear embeddings of (C, T') into E^3 which starts at f and ends at g (Theorem 4.2).

§§2 and 3 contain lemmas used in the proof of Theorem 4.2

DEFINITIONS. Let C be a complex with triangulation T. A *linear embedding* of C (or (C, T)) into E^n is an embedding that is linear on each simplex of T. A *linear isotopy* h_t: $(C, T) \to E^n$ ($t \in [0, 1]$) is a continuous family of linear embeddings.

2. **Linear isotopies on 2-spheres.** In dealing with a complex C linearly embedded in E^3, it will be convenient to consider the intersection of C with a small round 2-sphere centered at a vertex v of C. This intersection, $C \cap S$, is a natural embedding of $\mathrm{Lk}(v)$ into S. An embedding of a complex into S that can be so obtained is called a *spherically linear embedding*. A *spherically linear isotopy* is a continuous family of spherically linear embeddings.

Let π be the homeomorphism which takes the open lower hemisphere of a unit 2-sphere centered at (0, 0, 1) onto the $(x, y, 0)$ plane defined by taking each point p of the hemisphere to the point $(x, y, 0)$ which lies on the line determined by (0, 0, 1) and p. Note that π provides a one-to-one correspondence between segments of great circles in the hemisphere and straight line

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intervals in the plane. The symbol π will denote any such homeomorphism from a hemisphere of a round 2-sphere onto a plane.

The purpose of this section is to prove the following lemma.

LEMMA 2.1. *Let P be a disk with* Bd P *triangulated. Let h_t:* Bd $P \to S^2$ *($t \in [0, 1]$) be a spherically linear isotopy such that $h_0 = h_1$ and let D be a component of $S^2 - h_0(\text{Bd } P)$. Then there are a triangulation T of P which extends the given triangulation of* Bd P *and a spherically linear isotopy H_t: $(P, T) \to S^2$ ($t \in [0, 1]$) so that $H_0(P) = \bar{D}$, H_t extends h_t for each $t \in [0, 1]$, and $H_0 = H_1$.*

PROOF. First find a triangulation T_1 and a spherically linear isotopy H_t^1 ($t \in [0, 1]$) of (P, T_1) which satisfies the conclusions of the lemma except that H_0^1 may not equal H_1^1. This spherically linear isotopy can be obtained as follows. Let S^2 be a standard round 2-sphere in E^3 with center v, C be a PL disk in E^3 that is triangulated by coning from Bd C to an interior point w, and $\tilde{h}_t$: $C \to E^3$ ($t \in [0, 1]$) be a linear isotopy which witnesses the fact that h_t is a spherically linear isotopy, that is, for each t in $[0, 1]$, $\tilde{h}_t(w) = v$ and $\tilde{h}_t(C) \cap S^2 = h_t(\text{Bd } P)$. Extend $\tilde{h}_t$ to a linear isotopy $\tilde{H}_t$: $E^3 \to E^3$ ($t \in [0, 1]$) by [2, Theorem 3.3]. Let Σ be a small round 2-sphere in E^3 concentric with S^2 contained in $\tilde{H}_t(\text{St}(w))$ for each t in $[0, 1]$. The family of embeddings of Lk(w) into Σ described by the intersections $\tilde{H}_t(\text{St}(w)) \cap \Sigma$ ($t \in [0, 1]$) is a spherically linear isotopy of Lk(w) into Σ. By expanding Σ out to S^2 one obtains a corresponding spherically linear isotopy of Lk(w) into S^2. Restricting this spherically linear isotopy to the disk P of Lk(w) which maps onto $\bar{D}$ at time $t = 0$ yields a triangulation T_1 of P and a desired spherically linear isotopy H_t^1 ($t \in [0, 1]$) of (P, T_1) into S^2.

Next find a triangulation T refining T_1 and a spherically linear isotopy H_t^2 ($t \in [0, 1]$) of (P, T) such that $H_0^2 = H_1^1$ and $H_1^2|$(the 1-skeleton of T_1) $=$ $H_0^1|$(the 1-skeleton of T_1). The spherically linear isotopy H_t^2 ($t \in [0, 1]$) can be obtained by considering a shelling $\sigma_1, \sigma_2, \ldots, \sigma_n$ of the triangulation T_1 of P and moving the σ_i's into the desired positions one at a time. First subdivide P and produce a spherically linear isotopy which starts at H_1^1 and moves Bd σ_1 into the desired position leaving Bd P fixed. Next produce a spherically linear isotopy which begins with the homeomorphism at which the previous one ends, leaves Bd P fixed, leaves the map on σ_1 fixed, and at its conclusion maps Bd σ_2 as desired. Carrying on in this fashion through the σ_i's and finding a triangulation T with respect to which each of these isotopies is a spherically linear isotopy produces a desired spherically linear isotopy H_t^2 ($t \in [0, 1]$).

Let σ be a 2-simplex of T_1. For each such σ find an open hemisphere of S^2 which contains $H_0^1(\sigma)$ and use a map π to project that hemisphere onto the plane E^2. Then $\pi \circ H_0^1|\sigma$ and $\pi \circ H_1^2|\sigma$ are two linear embeddings of $(\sigma, T|\sigma)$ onto a triangle in E^2. By [2, Corollary 5.4], there is a linear isotopy of $(\sigma, T|\sigma)$ which starts at $\pi \circ H_1^2|\sigma$, ends at $\pi \circ H_0^1|\sigma$, and leaves the boundary fixed throughout. The map π^{-1} can be used to produce a spherically linear isotopy of $(\sigma, T|\sigma)$ which starts at $H_1^2|\sigma$, ends at $H_0^1|\sigma$, and leaves the boundary fixed. Let H_t^3 ($t \in [0, 1]$) be a spherically linear isotopy of (P, T) which performs the above moves on each such 2-simplex σ of T_1 simultaneously.

The desired spherically linear isotopy H_t ($t \in [0, 1]$) is essentially obtained by performing H_t^1, then H_t^2, and finally H_t^3. Slight modifications are necessary to ensure that for each $t \in [0, 1]$, $H_t|\mathrm{Bd}\, P = h_t$.

3. **Flexible blisters.** In this section we show how to extend a linear isotopy of a 2-manifold into E^3 to a blister over a submanifold.

DEFINITION. If D is a submanifold of a manifold M, then a *blister* over D is a set C homeomorphic to $D \times [0, 1]$ attached to M by identifying each point x of D with $(x, 0)$ in C.

THEOREM 3.1. *Let (M^2, T_M) be a compact, triangulated 2-manifold, D be an orientable submanifold, possibly with boundary, which is a subcomplex of T_M, and C be a blister over D. Let h_t: $(M^2, T_M) \to E^3$ ($t \in [0, 1]$) be a linear isotopy such that $h_0|D = h_1|D$ and let a side of each component of $h_0(D)$ be chosen. Then there is a linear isotopy H_t: $(M^2 \cup C, T) \to E^3$ ($t \in [0, 1]$) so that T extends T_M, $H_0(C)$ is on the chosen side, for each $t \in [0, 1]$, H_t extends h_t, and $H_0|C = H_1|C$.*

PROOF (simplified by R. H. Bing). We describe the extension H_t and thereby construct the image of the triangulated C rather than first describing the triangulation in the abstract.

Let ε be a positive number such that if σ and σ' are two simplexes of T_M such that σ has baricenter p and $p \notin \sigma'$, then $d(h_t(p), h_t(\sigma')) > \varepsilon$ for each t in $[0, 1]$. (Consider a vertex to be its own baricenter.)

Let σ^2 be a 2-simplex of D with baricenter p. Then for each $t \in [0, 1]$ a vertex v_p in C will be mapped by H_t to the point on the perpendicular to $h_t(\sigma^2)$ at $h_t(p)$ at distance $\varepsilon/2$ from $h_t(p)$ and on the correct side of $h_t(D)$. The cone from $H_t(v_p)$ down to $H_t(\sigma^2)$ is the image of a 3-simplex of C.

Let σ^1 be a 1-simplex of D whose interior is interior to D and having baricenter q. Then for each $t \in [0, 1]$ a vertex w in C will be mapped by H_t to the point at distance $\varepsilon/2$ from $h_t(q)$ which is on the correct side of $h_t(D)$ and on the bisector of the dihedral angle determined by the image under h_t of the two 2-simplexes of T_M having σ^1 as a common edge. Let p and r be the baricenters of those two 2-simplexes. The cone from $H_t(w)$ to the 2-simplex containing $H_t(v_p)$ and $H_t(\sigma^1)$ is a 3-simplex in the image of C as is the corresponding 3-simplex using $H_t(v_r)$ instead of $H_t(v_p)$. These 3-simplexes may require one subdivision later.

The blister has now been constructed over all of Int D except the vertices. Let v be an interior vertex of D. For each $t \in [0, 1]$, consider the round 2-sphere centered at $h_t(v)$ having radius $\varepsilon/2$. For brevity we denote this roving 2-sphere by S.

For each $t \in [0, 1]$, the intersection of S with the image under H_t of the part of C which has been described so far is an annulus. Let J_t be the boundary component of that annulus that does not lie on $h_t(M^2)$. Then J_t is a spherically linearly embedded simple closed curve. Let $\{x_i\}_{i=1}^k$ be the vertices of J_t. Note that x_i is the intersection of S with the image under H_t of a 1-simplex vu_i where $H_t(u_i)$ is near the baricenter of $h_t(\sigma)$ where σ is a 2-simplex or 1-simplex in $\mathrm{St}(v)$.

By Lemma 2.1 there is a spherically linear isotopy G_t ($t \in [0, 1]$) of a disk P into S so that for each $t \in [0, 1]$, $G_t(\text{Bd } P) = J_t$, J_t has only the vertices $\{x_i\}_{i=1}^k$, $G_t(P) \cap h_t(M^2) = \varnothing$, and $G_0 = G_1$. For each 2-simplex xyz of P, there is a 3-simplex $xyzv$ in the triangulation T of C. We define $H_t(x) = G_t(x)$, $H_t(y) = G_t(y)$, and $H_t(z) = G_t(z)$ for each t.

Now $H_t(C)$ has been constructed. It only remains to complete the triangulation of $H_t(C)$. Let σ^3 be a 3-simplex in $H_t(C)$ so that $\sigma^3 \cap h_t(M^2)$ is the image under h_t of an interior 1-simplex of D. Now Bd σ^3 may contain some extraneous 1-simplexes of the form $x_i x_{i+1}$ (count mod k). To complete a triangulation of Bd σ^3 let segments $x_i H(u_{i+1})$ be 1-simplexes. Finally, cone from the baricenter of σ^3 to Bd σ^3 to complete the triangulation of $H_t(C)$ and the proof of the lemma.

4. **Linear isotopies after subdivision.** In this section we prove that if a finite complex K is PL embedded in E^3 by f and g so that there is an orientation-preserving homeomorphism h of E^3 with $h \circ f = g$, then there is a triangulation T of K and a linear isotopy of (K, T) taking f to g. The result is a consequence of the following theorem.

THEOREM 4.1. *Let B^3 be a 3-ball with a shellable triangulation T. Let f and g be two linear embeddings of (B^3, T) into E^3 so that there is an orientation-preserving homeomorphism h of E^3 to itself with $h \circ f = g$. Then there is a subdivision T' of T and a linear isotopy H_t: $(B^3, T') \to E^3$ ($t \in [0, 1]$) such that $H_0 = f$ and $H_1 = g$.*

PROOF. The proof is by induction on n, the number of 3-simplexes in the triangulation T.

For $n = 1$, f and g are two linear embeddings of a 3-simplex with the same orientation. We can push f to g without any subdivision.

We assume that the theorem is true for triangulations with k 3-simplexes when k is less than n and assume that T has n 3-simplexes.

Let σ be a 3-simplex of T which can be removed first in a shelling of B^3. Let $T_1 = T|\text{Cl}(B^3 - \sigma)$. Then f and g restricted to $(\text{Cl}(B^3 - \sigma), T_1)$ meet the hypotheses of the theorem. By induction there exist a subdivision T_1' of T_1 and a linear isotopy H_t': $(\text{Cl}(B^3 - \sigma), T_1') \to E^3$ ($t \in [0, 1]$) so that $H_0' = f|\text{Cl}(B^3 - \sigma)$ and $H_1' = g|\text{Cl}(B^3 - \sigma)$.

The rest of the proof is devoted to extending H_t' to σ. The idea is to push $f(\sigma)$ down very close to $f(\text{Cl}(B^3 - \sigma))$, then perform H_t' with the squashed $f(\sigma)$ hanging on, and then expanding σ back out to where it belongs.

Let φ_t: $E^3 \to E^3$ ($t \in [0, 1]$) be a linear isotopy of E^3 so that $\varphi_0 = \text{id}$ and $\varphi_1 \circ g|\sigma = f|\sigma$.

Let G_t: $\text{Cl}(B^3 - \sigma) \to E^3$ ($t \in [0, 1]$) be the linear isotopy defined by

$$G_t = \begin{cases} H_{2t}' & \text{for } t \in \left[0, \frac{1}{2}\right], \\ \varphi_{2(t-1/2)} \circ H_1' & \text{for } t \in \left[1/2, 1\right]. \end{cases}$$

Let $D = \text{Cl}(B^3 - \sigma) \cap \sigma$. Note that $G_0|D = G_1|D$. Let C be a blister on D. Consider the 2-sphere which is $\text{Bd}(\text{Cl}(B^3 - \sigma))$. We are in a position to apply Lemma 3.1 to extend G_t to a linear isotopy G_t': $\text{Cl}(B^3 - \sigma) \cup C \to E^3$ ($t \in [0, 1]$) so that $G_0'|C = G_1'|C$.

Let λ_t: $B^3 \to E^3$ ($t \in [0, 1]$) be a linear isotopy such that $\lambda_0 = f$, for all $t \in [0, 1]$, $\lambda_t|B^3 - \sigma = f|B^3 - \sigma$, and $\lambda_1(\sigma) \subset G_0'(C)$. Thus λ_t pushes $f(\sigma)$ into the blister. Note that λ_t is constant except on σ.

Now we can define the entire linear isotopy of B^3. It consists of performing several linear isotopies one after the other. First perform λ_t (pushing σ into the blister). Second perform G_t' (pushing $\mathrm{Cl}(B^3 - \sigma)$ first to where it should be and then back to the position where the blister is where it started). Third, perform λ'_{1-t} ($t \in [0, 1]$) where this is defined to be the linear isotopy of B^3 that pushes σ exactly like λ_{1-t} ($t \in [0, 1]$) does and leaves $\mathrm{Cl}(B^3 - \sigma)$ constantly in the position it was after the first two linear isotopies, λ_t and G_t'. (This linear isotopy pulls σ back out to where it was originally under f.) Finally perform φ_{1-t} ($t \in [0, 1]$) composed with the embedding of B^3 determined by the final position of the first three isotopies. This composition is a required linear isotopy of B^3.

THEOREM 4.2. *Let K be a finite complex and let f and g be two PL embeddings of K into E^3 such that there is an orientation-preserving homeomorphism h of E^3 such that $h \circ f = g$. Then there is a triangulation T of K and a linear isotopy h_t: $(K, T) \to E^3$ ($t \in [0, 1]$) so that $h_0 = f$ and $h_1 = g$.*

PROOF. By [1, Theorem 9], there is an orientation-preserving PL homeomorphism h' of E^3 such that $h' \circ f = g$. So we can find a PL ball B which contains $f(K)$ and triangulates B with a triangulation T' that contains $f(K)$ as a subcomplex, is shellable, and has the property that $h'|B$ is linear with respect to T'. Now we can apply Theorem 4.1 to (B, T') where the two embeddings are the identity and $h'|B$, proving Theorem 4.2. □

Improved versions of Lemmas 2.1 and 3.1 and a variation of Theorem 4.2 will appear in future papers. The variation of Theorem 4.2 states that if K is a 3-cell and $f|\mathrm{Bd}\, K = g|\mathrm{Bd}\, K$, then the linear isotopy h_t ($t \in [0, 1]$) of the conclusion can be chosen to leave Bd K fixed.

Question. Is Theorem 4.2 true when "E^3" is replaced by "E^n"?

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