PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 65, Number 2, August 1977

CONTINUOUS HOMOMORPHISMS ARE DIFFERENTIABLE

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ABSTRACT. Suppose X is a Banach space, D is an open set of X containing 0, and V is a continuously differentiable function from DXD to X satisfying $V(0, x) = V(x, 0) = x$ for each x in D. If T is a continuous function from $[0, 1]$ into D satisfying $T(0) = 0$ and $V(T(s), T(t)) = T(s + t)$ whenever each of s, t, and $s + t$ is in $[0, 1]$ then T is continuously differentiable on $[0, 1]$.

Suppose X is a Banach space, D is an open convex set of X containing 0, and V is a continuously differentiable function from DXD to X satisfying $V(x, 0) = V(0, x) = x$ for each x in D.

THEOREM 1. *If T is a continuous function from* $[0, 1]$ *into D satisfying $T(0) = 0$ and $V(T(s), T(t)) = T(s + t)$ whenever each of s, t, and $s + t$ is in* $[0, 1]$ *then T is continuously differentiable on* $[0, 1]$.

Before giving a proof of Theorem 1 we will indicate some background and questions.

Suppose x is in D. Let $x^0 = 0$ and if n is a positive integer so that x^{n-1} exists and is in D let $x^n = V(x, x^{n-1})$. V is said to be power associative if for each x in D, $V(x^n, x^m) = x^{n+m}$ whenever each of x^n and x^m is in D and x^{n+m} exists.

If d is a positive number let $R(d) = \{x \text{ in } X \mid \|x\| < d\}$.

THEOREM 2. *If X and V are as above then there is a neighborhood E of* 0 *in X so that the restriction of V to EXE is power associative if and only if there are positive numbers r and d so that for each x in $R(r)$ there is a unique continuous function T_x from* $[0, 1]$ *to $R(d)$ satisfying $T_x(0) = 0$, $T_x(1) = 1$, and $V(T_x(s), T_x(t)) = T_x(s + t)$ whenever each of s, t, and $s + t$ is in* $[0, 1]$.

The nontrivial part of Theorem 2 was proved in [1]. Theorem 1 allows us to change " continuous " to "continuously differentiable" in Theorem 2.

F. A. Roach in [2] has recently given an interesting example of a power associative V where $X = D =$ Euclidean n dimensional space and $(1, 0, \ldots, 0)$ is the unit instead of $(0, \ldots, 0)$. If $n = 1, 2$, or 4 this V is real,

Received by the editors November 18, 1976.

AMS (MOS) subject classifications (1970). Primary 47H15, 22A99.

Key words and phrases. Differentiable homomorphism, automatic differentiability, one parameter semigroups.

complex, or quaternion multiplication. In other dimensions it is nonassociative. In each dimension V may be restricted to the unit sphere to provide a global power associative multiplication with unit there.

In view of Theorems 1 and 2 some questions arise concerning power associative multiplications. Suppose that V is power associative and choose r as in Theorem 2. Define f from $R(r)$ into X by $f(x) = T_x'(0)$. It is easy to see that $f(T_x(t)) = tf(x)$ for $\|x\|$ sufficiently small. If V is associative and continuously differentiable arguments from Birkhoff [3] can be used to show the restriction of f to some neighborhood of 0 is a homeomorphism. If V is twice differentiable then the argument in Neuberger's paper [4] shows that f is continuously differentiable on some neighborhood of 0.

Questions. Must f be continuously differentiable? Must the restriction of f to some neighborhood of 0 be a homeomorphism? Is the answer to either of these yes in the finite dimensional case?

An affirmative to either of these would allow one to use f to construct a multiplication W locally isomorphic with V at 0 so that $W(tx, sx) = (s + t)x$ whenever each of s, t, and $s + t$ is in $[0, 1]$ and $\|x\|$ is sufficiently small.

We now construct a proof of Theorem 1. If G is a continuous linear transformation from XxX to X or from X to X denote the norm of G by $|G|$.

LEMMA 1. *If V and T are as above there is a positive number M so that if each of s and t is in* $[0, 1]$ *then* $\|T(s) - T(t)\| < M|s - t|$.

PROOF. Suppose each of t, e, and $t + e$ is in $[0, 1]$. $\int_0^{t+e} ds T(s) = \int_0^t ds T(s) + \int_0^e ds T(t + s)$. Thus $\int_0^e ds[T(t + s) - T(s)] = \int_0^t ds[T(e + s) - T(s)]$. If t is in $[0, 1)$ and e is in $(0, 1 - t]$ denote by $A(t, e)$ the point $e^{-1}\int_0^e ds[T(t + s) - T(s)]$. By the fundamental theorem of calculus, $A(t, e)$ has limit $T(t)$ as e approaches 0 from the right.

Note

$$A(t, e) = \frac{1}{e}\int_0^t ds\big[V(T(s), T(e)) - V(T(s), 0)\big]$$
$$= \frac{1}{e}\int_0^t ds\int_0^1 dr V'(T(s), rT(e))(0, T(e)).$$

An easy calculation shows $V'(0, 0)(a, b) = a + b$ whenever each of a and b is in X. Hence, if t is in $[0, 1]$ then

$$\int_0^t ds\int_0^1 dr V'(0, 0)\Big(0, \frac{1}{e}T(e)\Big) = \frac{t}{e}T(e).$$

Choose the positive number d so that if each of a and b is in $R(d)$ then $|V'(a, b) - V'(0, 0)| < \frac{1}{2}$. Choose the positive number d' so that if t is in $[0, d']$ then $\|T(t)\| < d$. Choose the positive number d'' so that if e is in $(0, d'')$ then $\|T(d') - A(d', e)\| < \frac{1}{2}$.

Suppose e is in $(0, d'')$. Then

$$\left\|T(d') - \frac{d'}{e}T(e)\right\| < \|T(d') - A(d', e)\| + \left\|A(d', e) - \frac{d'}{e}T(e)\right\|$$

$$\leqslant \frac{1}{2} + \left\|\int_0^{d'} ds \int_0^1 dr\left[V'(T(s), rT(e)) - V'(0, 0)\right]\left(0, \frac{1}{e}T(e)\right)\right\|$$

$$\leqslant \frac{1}{2} + \frac{1}{2}\frac{d'}{e}\|T(e)\|$$

since for each s in $[0, d']$ and each r in $[0, 1]$ each of $T(s)$ and $rT(e)$ is in $R(d)$.

Let B denote the number $(2/d')[\frac{1}{2} + \|T(d')\|]$. We have, if e is in $(0, d'')$,

$$\left\|\frac{d'}{e}T(e)\right\| - \|T(d')\| \leqslant \left\|T(d') - \frac{d'}{e}T(e)\right\| \leqslant \frac{1}{2} + \frac{1}{2}\left\|\frac{d'}{e}T(e)\right\|.$$

Hence $\|e^{-1}T(e)\| < B$.

By continuity of T and V' and compactness of $[0, 1]$ choose the number W so that if each of t, s, and e is in $[0, 1]$ then $|V'(T(s), tT(e))| < W$. Let C be the maximum of $\{\|T(t)\|: t \text{ is in } [0, 1]\}$.

Suppose each of t and e is in $[0, 1]$ and $e < t$.

$$\|T(t) - T(e)\| = \|V(T(e), T(t - e)) - V(T(e), 0)\|$$

$$= \left\|\int_0^1 ds V'(T(e), sT(t - e))(0, T(t - e))\right\| \leqslant W\|T(t - e)\|.$$

Thus there are two alternatives. If $t - e$ is in $(0, d'')$ then $W\|T(t - e)\| \leqslant WB|t - e|$. If $t - e$ is in $[d'', 1]$ then $W\|T(t - e)\| \leqslant (WC/d'')|t - e|$. Thus, the choice of $M = \max\{WB, WC/d''\}$ satisfies the conclusion of Lemma 1.

LEMMA 2. *If c is a positive number there is a positive number m so that if e is in* $[0, 1]$ *and x is in $R(m)$ then*

$$\|V(T(e), x) - V(T(e), 0) - V'(T(e), 0)(0, x)\| < c\|x\|.$$

PROOF. Let $L(X)$ denote the Banach space of continuous linear transformations from X to X and define the function F from $[0, 1]$ to $L(X)$ by $F(s)(y) = V'(T(s), 0)(0, y)$ for each y in X. F is continuous.

Choose d and d' as in the proof of Lemma 1 and denote by G the function from $[0, d']$ to $L(X)$ defined by $G(t) = \int_0^t dsF(s)$. Denote by I the identity transformation in $L(X)$. If t is in $(0, d']$ then

$$\left|\frac{1}{t}G(t) - I\right| = \sup_{\|y\|=1}\left\|\frac{1}{t}\int_0^t dsV'(T(s), 0) - V'(0, 0)(0, y)\right\| \leqslant \frac{1}{2}$$

since $\|y\| = 1$ and for each s in $[0, t]$ $T(s)$ is in $R(d)$. Thus, $G(t)$ is invertible.

Suppose c is a positive number and for each s in $[0, 1]$ choose the positive number d_s so that if each of $T(s) - p$ and q is in $R(d_s)$ then $|V'(T(s), 0) - V'(p, q)| < c/2$.

By continuity of T choose d_s' positive so that if e is in $[0, 1]$ and $|e - s| < d_s'$ then $T(e) - T(s)$ is in $R(d_s)$. By compactness of $[0, 1]$ there is a finite set H in $[0, 1]$ so that if e is in $[0, 1]$ there is an s in H so that $|e - s| < d_s'$. Choose such an H and let m be the smallest member of $\{d_s: s \text{ is in } H\}$.

Suppose e is in $[0, 1]$ and x is in $R(m)$. Choose s in H so that $|e - s| < d_s'$. We then have

$$\|V(T(e), x) - V(T(e), 0) - V'(T(e), 0)(0, x)\|$$
$$\leq \left\| \int_0^1 dr\left[V'(T(e), rx) - V'(T(s), 0)\right](0, x) \right\|$$
$$+ \left\| \left[V'(T(s), 0) - V'(T(e), 0)\right](0, x) \right\|.$$

But for each r in $[0, 1]$ rx is in $R(m)$ which is contained in $R(d_s)$ and $T(e) - T(s)$ is in $R(d_s)$. Hence the right-hand side of this last inequality is less than $c\|x\|$.

Thus we have a proof of Lemma 2.

Choose M as in the conclusion of Lemma 1. Suppose c is a positive number and choose t in $(0, d')$. Choose m positive by Lemma 2 so that if x is in $R(m)$ then $\|V(T(s), x) - V(T(s), 0) - V'(T(s), 0)(0, x)\| < (c/2M|G(t)^{-1}|)\|x\|$ for each s in $[0, 1]$. Choose n positive so that if e is in $(0, n)$ then $T(e)$ is in $R(m)$.

Suppose s is in $[0, 1]$ and e is in $(0, n)$. Then

$$\|T(s + e) - T(s) - F(s)(T(e))\|$$
$$= \|V(T(s), T(e)) - V(T(s), 0) - V'(T(s), 0)(0, T(e))\|$$
$$< \left(c/2M|G(t)^{-1}|\right)\|T(e)\| \leq \left(c/2|G(t)^{-1}|\right)e.$$

Thus,

$$\frac{1}{e}\left\| \int_0^t ds\left(T(s + e) - T(s) - F(s)(T(e))\right) \right\|$$
$$= \left\| A(t, e) - G(t)\left(\frac{1}{e} T(E)\right) \right\| < \left(c/2|G(t)^{-1}|\right).$$

Choose n' positive so that if e is in $(0, n')$ then $\|T(t) - A(t, e)\| < (c/2|G(t)^{-1}|)$. If e is in both of $(0, n)$ and $(0, n')$ we may combine the last two inequalities to obtain $\|T(t) - G(t)(e^{-1}T(e))\| < (c/|G(t)^{-1}|)$ and finally $\|G(t)^{-1}(T(t)) - e^{-1}T(e)\| < c$.

Thus the limit of $e^{-1}T(e)$ as e approaches 0 from the right exists and is $G(t)^{-1}(T(t))$. Denote this limit by $T'(0)$. We have for each t in $(0, d')$, $T(t) = G(t)(T'(0))$. Hence, by the fundamental theorem of calculus T is continuously differentiable on $[0, d')$.

Suppose t is in $[0, 1]$ and T is continuously differentiable on $[0, t)$. $T(s) =$

$V(T(s/2), T(s/2))$ so by the chain rule it follows that T is continuously differentiable on $[0, \min(2t, 1))$. It follows that T is continuously differentiable on $[0, 1]$.

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