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ON THE INNOVATION THEOREM

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ABSTRACT. Let $z(t)$, $0 < t < T$, be the signal process and $y(t) = \int_0^t z(r)\, dr + w(t)$ be the observation process where $w(t)$ is a process of independent increments. It is shown that, under certain conditions, the innovation process $v(t) = y(t) - \int_0^t E(z(r)|y(u), 0 < u < r)\, dr$, has the same probability law as $w(t)$.

1. **Introduction**. Let $z(t)$, $0 \leqslant t \leqslant T$, be a (t, ω)-measurable signal process in R^m such that

$$\int_0^T E|z(t)|^2\, dt < \infty, \tag{1.1}$$

where $|\cdot|$ denotes the m-dimensional vector norm as well as the absolute value of a number, and let

$$y(t) = \int_0^t z(s)\, ds + w(t),$$

for $0 < t < T$, be the observation process where $w(t)$ is an m-dimensional Wiener process such that

$$\begin{aligned} &W(t) - w(s) \text{ is independent of} \\ &\mathcal{G}_s = \sigma\{z(r), w(r); 0 \leqslant r \leqslant s\} \text{ for } 0 \leqslant s \leqslant t \leqslant T. \end{aligned} \tag{1.2}$$

Let $\mathcal{F}_s = \sigma\{y(r)\colon 0 < r \leqslant s\}$, $0 \leqslant s \leqslant T$. Condition (1.1) implies the existence of $E\{z(s)|\mathcal{F}_s\}$ which is measurable in (s, ω). Indeed, this condition can be weakened somewhat (see [3, p. 21 and Remark, p. 23]). The process

$$v(t) = y(t) - \int_0^t E\{z(s)|\mathcal{F}_s\}\, ds$$

is called the innovation process. Application of the innovation processes to optimal filtering can be found in [1], [2], [4], [6], [7].

It has been known that $\{v(t), \mathcal{F}_t\}$ is an L^2-martingale with the same quadratic variation as $w(t)$. This result was extended to the case that $w(t)$ is an L^2-martingale with conditional independent increments w.r.t. $\{\mathcal{G}_t\}$ (see [5, Theorem 3]). If $w(t)$ is the Wiener process again, the continuity of $v(t)$

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implies that $v(t)$ is also a Wiener process (see [3, Lemma 2.2], [5, Theorem 2]). These results are the innovation theorems. Combining the above results, it is naturally interesting to know whether $v(t)$ has the same probability law as $w(t)$ if $w(t)$ is a process of independent increments. This paper shows that this is true, at least under certain conditions.

2. **Generalization.** Let all the notations be defined as in §1 except that $w(t)$ is now a process of independent increments satisfying condition (1.2) and

$$Ew(t) = 0, \quad E|w(t) - w(s)|^2 \leqslant c^2(t - s), \tag{2.1}$$

for $0 \leqslant s \leqslant t \leqslant T$ where $c \in R^+$. Let $x(t) = z(t) - E\{z(t)|\mathcal{F}_t\}$, $0 \leqslant t \leqslant T$. Then

$$v(t) = \int_0^t x(s)\,ds + w(t).$$

It is easy to see that $\{v(t), \mathcal{F}_t\}$ is a martingale.

For $0 \leqslant s < t \leqslant T$, let $t_k = s + k(t - s)2^{-n}$, $0 \leqslant k \leqslant 2^n$. Also, for $0 \leqslant k < 2^n$, let

$$\Delta v(t_k) = v(t_{k+1}) - v(t_k), \qquad \Delta w(t_k) = w(t_{k+1}) - w(t_k)$$

and

$$\Delta I_k(x) = \int_{t_k}^{t_{k+1}} x(r)\,dr.$$

LEMMA 1. (a) $\lim_{n\to\infty}\Sigma_k E\{|\Delta I_k(x)|^2\} = 0$.
(b) $\lim_{n\to\infty}\Sigma_k E\{|\Delta w(t_k)|\,|\Delta I_k(x)|\} = 0$.

PROOF. By Schwarz's inequality and Fubini's theorem,

$$E\left\{|\Delta I_k(x)|^2\right\} \leqslant (t_{k+1} - t_k)\int_{t_k}^{t_{k+1}} E|x(s)|^2\,ds, \tag{2.2}$$

$$\begin{aligned} E\left\{|\Delta w(t_k)||\Delta I_k(x)|\right\} &\leqslant \left\{E|\Delta w(t_k)|^2 E|\Delta I_k(x)|^2\right\}^{1/2} \\ &\leqslant c(t_{k+1} - t_k)\left\{\int_{t_k}^{t_{k+1}} E|x(s)|^2\,ds\right\}^{1/2}. \end{aligned} \tag{2.3}$$

In deriving the last inequality, we have used both (2.1) and (2.2). Since

$$E|x(s)|^2 \leqslant E|x(s)|^2 + E|E\{z(s)|\mathcal{F}_s\}|^2 = E|z(s)|^2,$$

(a) follows from (2.2) and (1.1). From (2.3),

$$\sum_k E\left\{|\Delta w(t_k)||\Delta I_k(x)|\right\} \leqslant c(t - s)\max_k\left\{\int_{t_k}^{t_{k+1}} E|z(s)|^2\,ds\right\}^{1/2}.$$

Therefore, (b) follows from the above inequality and (1.1).

For $0 \leqslant k \leqslant 2^n$, let

$$f_k = e^{i\lambda\cdot[w(t) - w(t_k) + v(t_k) - v(s)]},$$

where $\lambda \in R^m$ and where "$\cdot$" denotes the dot product in R^m.

LEMMA 2. *For* $0 \leq k < 2^n$,

$$E(f_{k+1} - f_k | \mathfrak{F}_s) = E\{f_k e^{-i\lambda\cdot\Delta w(t_k)}(J_1 + J_2)(k) | \mathfrak{F}_s\} \tag{2.4}$$

with

$$|J_1(k)| \leq \tfrac{1}{2}|\lambda|^2|\Delta I_k(x)|^2, \quad |J_2(k)| \leq |\lambda|^2|\Delta w(t_k)||\Delta I_k(x)|. \tag{2.5}$$

PROOF. Note that if we write

$$e^{iu} = 1 + R_1(u) = 1 + iu + R_2(u), \tag{2.6}$$

then

$$|R_1(u)| \leq |u|, \quad |R_2(u)| \leq \tfrac{1}{2}|u|^2. \tag{2.7}$$

Now since $\Delta v(t_k) = \Delta I_k(x) + \Delta w(t_k)$, it is easy to check that

$$f_{k+1} - f_k = f_k(e^{i\lambda\cdot\Delta I_k(x)} - 1).$$

Therefore, by using (2.6),

$$\begin{aligned} E(f_{k+1} - f_k | \mathfrak{F}_s) &= E\{f_k(e^{i\lambda\cdot\Delta I_k(x)} - 1) | \mathfrak{F}_s\} \\ &= E\{f_k[i\lambda \cdot \Delta I_k(x) + R_2(\lambda \cdot \Delta I_k(x))] | \mathfrak{F}_s\} \\ &= E\{f_k e^{-i\lambda\cdot\Delta w(t_k)} e^{i\lambda\cdot\Delta w(t_k)} i\lambda \cdot \Delta I_k(x) | \mathfrak{F}_s\} \\ &\quad + E\{f_k R_2(\lambda \cdot \Delta I_k(x)) | \mathfrak{F}_s\} \\ &= E\{f_k e^{-i\lambda\cdot\Delta w(t_k)}[1 + R_1(\lambda \cdot \Delta w(t_k))] i\lambda \cdot \Delta I_k(x) | \mathfrak{F}_s\} \\ &\quad + E\{f_k e^{-i\lambda\cdot\Delta w(t_k)} e^{i\lambda\cdot\Delta w(t_k)} R_2(\lambda \cdot \Delta I_k(x)) | \mathfrak{F}_s\} \\ &= E\{f_k e^{-i\lambda\cdot\Delta w(t_k)}[i\lambda \cdot \Delta I_k(x) + (J_1 + J_2)(k)] | \mathfrak{F}_s\}, \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} J_1(k) &= e^{i\lambda\cdot\Delta w(t_k)} R_2(\lambda \cdot \Delta I_k(x)), \\ J_2(k) &= i\lambda \cdot \Delta I_k(x) R_1(\lambda \cdot \Delta w(t_k)). \end{aligned} \tag{2.9}$$

By using the fact that $\mathfrak{F}_t \subset \mathcal{G}_t$ and assumption (1.2),

$$\begin{aligned} &E\{f_k e^{-i\lambda\cdot\Delta w(t_k)} \lambda \cdot \Delta I_k(x) | \mathfrak{F}_s\} \\ &\quad = E\left\{E\left(e^{i\lambda\cdot[w(t) - w(t_{k+1}) + v(t_k) - v(s)]}[\lambda \cdot \Delta I_k(x)] | \mathcal{G}_{t_{k+1}}\right) | \mathfrak{F}_s\right\} \\ &\quad = Ee^{i\lambda\cdot[w(t) - w(t_{k+1})]} E\{e^{i\lambda\cdot[v(t_k) - v(s)]}[\lambda \cdot \Delta I_k(x)] | \mathfrak{F}_s\} \\ &\quad = Ee^{i\lambda\cdot[w(t) - w(t_{k+1})]} E\{e^{i\lambda\cdot[v(t_k) - v(s)]} E(\lambda \cdot \Delta I_k(x) | \mathfrak{F}_{t_k}) | \mathfrak{F}_s\} \\ &\quad = 0. \end{aligned} \tag{2.10}$$

The last inequality follows from the facts that $\Delta I_k(x) = \Delta v(t_k) - \Delta w(t_k)$, $\{v(t), \mathfrak{F}_t\}$ is a martingale, and assumptions (1.2) and (2.1). Therefore, (2.8) and (2.10) imply equality (2.4). Inequality (2.5) follows from (2.9), (2.7) and the fact that $|e^{i\lambda\cdot w(t_k)}| = 1$. This completes the proof of Lemma 2.

THEOREM 3. *For* $0 \leqslant s < t \leqslant T$,

$$E\left\{e^{i\lambda\cdot[v(t)-v(s)]}\middle|\mathcal{F}_s\right\} = Ee^{i\lambda\cdot[w(t)-w(s)]}.$$

PROOF. By Lemma 2,

$$\begin{aligned} &E\left\{e^{i\lambda\cdot[v(t)-v(s)]} - e^{i\lambda\cdot[w(t)-w(s)]}\middle|\mathcal{F}_s\right\} \\ &\quad = E(f_{2^n} - f_0|\mathcal{F}_s) = \sum_k E(f_{k+1} - f_k|\mathcal{F}_s) \\ &\quad = \sum_k E\left\{f_k e^{-i\lambda\cdot\Delta w(t_k)}(J_1 + J_2)(k)\middle|\mathcal{F}_s\right\}. \end{aligned} \tag{2.11}$$

From the fact that $|f_k e^{-i\lambda\cdot\Delta w(t_k)}| \leqslant 1$, and from Lemma 1, it is easily seen that

$$\lim_{n\to\infty} \sum_k E\left|f_k e^{-i\lambda\cdot\Delta w(t_k)}(J_1 + J_2)(k)\right| = 0.$$

Therefore, by letting $n \to \infty$ in (2.11), we have

$$E\left\{e^{i\lambda\cdot[v(t)-v(s)]} - e^{i\lambda\cdot[w(t)-w(s)]}\middle|\mathcal{F}_s\right\} = 0.$$

Since $\mathcal{F}_s \subset \mathcal{G}_s$, Theorem 3 follows from the above identity and from assumption (1.2).

COROLLARY 4. *The innovation process* $\{v(t), \mathcal{F}_t\}$, $0 \leqslant t \leqslant T$, *is a process of independent increments with the same probability law as* $w(t)$.

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