ON THE INNOVATION THEOREM

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Abstract. Let \( z(t), 0 < t < T, \) be the signal process and \( y(t) = \int_0^t z(r) \, dr + w(t) \) be the observation process where \( w(t) \) is a process of independent increments. It is shown that, under certain conditions, the innovation process \( v(t) = y(t) - \int_0^t E(z(r)|y(u), 0 < u < r) \, dr, \) has the same probability law as \( w(t) \).

1. Introduction. Let \( z(t), 0 < t < T, \) be a \((t, \omega)\)-measurable signal process in \( \mathbb{R}^m \) such that

\[
\int_0^T E|z(t)|^2 \, dt < \infty,
\]

where \( | \cdot | \) denotes the \( m \)-dimensional vector norm as well as the absolute value of a number, and let

\[
y(t) = \int_0^t z(s) \, ds + w(t),
\]

for \( 0 < t < T, \) be the observation process where \( w(t) \) is an \( m \)-dimensional Wiener process such that

\[
W(t) - w(s) \text{ is independent of } \mathcal{G}_s = \sigma\{z(r), w(r); 0 \leq r < s\} \text{ for } 0 < s < t < T.
\]

Let \( \mathcal{G}_s = \sigma\{y(r); 0 < r < s\}, 0 < s < T. \) Condition (1.1) implies the existence of \( E\{z(s)|\mathcal{G}_s\} \) which is measurable in \((s, \omega)\). Indeed, this condition can be weakened somewhat (see [3, p. 21 and Remark, p. 23]). The process

\[
v(t) = y(t) - \int_0^t E\{z(s)|\mathcal{G}_s\} \, ds
\]

is called the innovation process. Application of the innovation processes to optimal filtering can be found in [1], [2], [4], [6], [7].

It has been known that \( \{v(t), \mathcal{G}_t\} \) is an \( L^2 \)-martingale with the same quadratic variation as \( w(t) \). This result was extended to the case that \( w(t) \) is an \( L^2 \)-martingale with conditional independent increments w.r.t. \( \{\mathcal{G}_t\} \) (see [5, Theorem 3]). If \( w(t) \) is the Wiener process again, the continuity of \( v(t) \).
implies that \( v(t) \) is also a Wiener process (see [3, Lemma 2.2], [5, Theorem 2]).

These results are the innovation theorems. Combining the above results, it is
naturally interesting to know whether \( v(t) \) has the same probability law as
\( w(t) \) if \( w(t) \) is a process of independent increments. This paper shows that this
is true, at least under certain conditions.

2. Generalization. Let all the notations be defined as in §1 except that \( w(t) \)
is now a process of independent increments satisfying condition (1.2) and

\[
Ew(t) = 0, \quad E[w(t) - w(s)]^2 \leq c^2(t - s),
\]

for \( 0 < s < t < T \) where \( c \in \mathbb{R}^+ \). Let \( x(t) = z(t) - E\{z(t)|\mathcal{F}_t\}, 0 < t < T \). Then

\[
v(t) = \int_0^t x(s) \, ds + w(t).
\]

It is easy to see that \( \{v(t), \mathcal{F}_t\} \) is a martingale.

For \( 0 < s < t < T \), let \( t_k = s + k(t - s)2^{-n}, 0 < k < 2^n \). Also, for \( 0 < k < 2^n \), let

\[
\Delta v(t_k) = v(t_{k+1}) - v(t_k), \quad \Delta w(t_k) = w(t_{k+1}) - w(t_k)
\]

and

\[
\Delta I_k(x) = \int_{t_k}^{t_{k+1}} x(r) \, dr.
\]

Lemma 1. (a) \( \lim_{n \to \infty} \sum_k E\{[\Delta I_k(x)]^2\} = 0 \).

(b) \( \lim_{n \to \infty} \sum_k E\{[\Delta w(t_k)] [\Delta I_k(x)]\} = 0 \).

Proof. By Schwarz’s inequality and Fubini’s theorem,

\[
E\{[\Delta I_k(x)]^2\} < (t_{k+1} - t_k) \int_{t_k}^{t_{k+1}} E|x(s)|^2 \, ds,
\]

\[
E\{[\Delta w(t_k)][\Delta I_k(x)]\} \leq \left( E[\Delta w(t_k)]^2 E[\Delta I_k(x)]^2 \right)^{1/2}
\]

\[
< c(t_{k+1} - t_k) \left( \int_{t_k}^{t_{k+1}} E|x(s)|^2 \, ds \right)^{1/2}.
\]

In deriving the last inequality, we have used both (2.1) and (2.2). Since

\[
E|x(s)|^2 < E|x(s)|^2 + E[E\{z(s)|\mathcal{F}_s\}]^2 = E|z(s)|^2,
\]

(a) follows from (2.2) and (1.1). From (2.3),

\[
\sum_k E\{[\Delta w(t_k)][\Delta I_k(x)]\} < c(t - s) \max_k \left\{ \int_{t_k}^{t_{k+1}} E|z(s)|^2 \, ds \right\}^{1/2}.
\]

Therefore, (b) follows from the above inequality and (1.1).

For \( 0 < k < 2^n \), let

\[
f_k = \exp\{w(t) - w(s) + v(t_k) - v(s)\},
\]
where $\lambda \in \mathbb{R}^m$ and where "•" denotes the dot product in $\mathbb{R}^m$.

**Lemma 2.** For $0 < k < 2^n$,

\begin{equation}
E (f_{k+1} - f_k | \mathcal{F}_k) = E \left\{ f_k e^{-\lambda \cdot \Delta w(t_k)} (J_1 + J_2)(k) | \mathcal{F}_k \right\}
\end{equation}

with

\begin{equation}
|J_1(k)| < \frac{1}{2} |\lambda|^2 |\Delta I_k(x)|^2, \quad |J_2(k)| < |\lambda|^2 |\Delta w(t_k)| |\Delta I_k(x)|.
\end{equation}

**Proof.** Note that if we write

\begin{equation}
e^{iu} = 1 + R_1(u) = 1 + iu + R_2(u),
\end{equation}

then

\begin{equation}
|R_1(u)| \leq |u|, \quad |R_2(u)| \leq \frac{1}{2} |u|^2.
\end{equation}

Now since $\Delta v(t_k) = \Delta I_k(x) + \Delta w(t_k)$, it is easy to check that

\begin{equation}
f_{k+1} - f_k = f_k (e^{i\lambda \Delta I_k(x)} - 1).
\end{equation}

Therefore, by using (2.6),

\begin{equation}
E (f_{k+1} - f_k | \mathcal{F}_k) = E \left\{ f_k (e^{i\lambda \Delta I_k(x)} - 1) | \mathcal{F}_k \right\}
\end{equation}

\begin{align*}
&= E \left\{ f_k [i\lambda \cdot \Delta I_k(x) + R_2(\lambda \cdot \Delta I_k(x))] | \mathcal{F}_k \right\} \\
&= E \left\{ f_k e^{-\lambda \cdot \Delta w(t_k)} e^{i\lambda \cdot \Delta I_k(x)} i\lambda \cdot \Delta I_k(x) | \mathcal{F}_k \right\} \\
&\quad + E \left\{ f_k R_2(\lambda \cdot \Delta I_k(x)) | \mathcal{F}_k \right\} \\
&= E \left\{ f_k e^{-\lambda \cdot \Delta w(t_k)} [1 + R_1(\lambda \cdot \Delta w(t_k))] i\lambda \cdot \Delta I_k(x) | \mathcal{F}_k \right\} \\
&\quad + E \left\{ f_k e^{-\lambda \cdot \Delta w(t_k)} e^{i\lambda \cdot \Delta I_k(x)} R_2(\lambda \cdot \Delta I_k(x)) | \mathcal{F}_k \right\} \\
&= E \left\{ f_k e^{-\lambda \cdot \Delta w(t_k)} i\lambda \cdot \Delta I_k(x) + (J_1 + J_2)(k) | \mathcal{F}_k \right\},
\end{align*}

where

\begin{align*}
J_1(k) &= e^{i\lambda \Delta w(t_k)} R_2(\lambda \cdot \Delta I_k(x)), \\
J_2(k) &= i\lambda \cdot \Delta I_k(x) R_1(\lambda \cdot \Delta w(t_k)).
\end{align*}

By using the fact that $\mathcal{F}_t \subset \mathcal{F}_s$, and assumption (1.2),

\begin{align*}
E \left\{ f_k e^{-\lambda \cdot \Delta w(t_k)} \lambda \cdot \Delta I_k(x) | \mathcal{F}_s \right\} \\
&= E \left\{ E \left( e^{i\lambda \cdot \Delta w(t_k)} \lambda \cdot \Delta I_k(x) | \mathcal{F}_s \right) | \mathcal{F}_s \right\} \\
&= E \left( e^{i\lambda \cdot \Delta w(t_k)} \lambda \cdot \Delta I_k(x) | \mathcal{F}_s \right)
\end{align*}

\begin{equation}
\text{(2.10)}
= 0.
\end{equation}

The last inequality follows from the facts that $\Delta I_k(x) = \Delta v(t_k) - \Delta w(t_k)$, $\{v(t), \mathcal{F}_t\}$ is a martingale, and assumptions (1.2) and (2.1). Therefore, (2.8) and (2.10) imply equality (2.4). Inequality (2.5) follows from (2.9), (2.7)
and the fact that $|e^{i\lambda w(t)}| = 1$. This completes the proof of Lemma 2.

**Theorem 3.** For $0 < s < t < T,$

$$E \left\{ e^{i\lambda \{v(t) - v(s)\}} \left| \mathcal{F}_s \right. \right\} = E e^{i\lambda \{w(t) - w(s)\}}.$$

**Proof.** By Lemma 2,

$$E \left\{ e^{i\lambda \{v(t) - v(s)\}} - e^{i\lambda \{w(t) - w(s)\}} \left| \mathcal{F}_s \right. \right\} = E (f_{2^n} - f_0) \left| \mathcal{F}_s \right. = \sum_k E (f_{k+1} - f_k) \left| \mathcal{F}_s \right.\left| f_{2^n} - f_0 \right| \left| \mathcal{F}_s \right.$$

$$= \sum_k E \left\{ f_k e^{-i\lambda \Delta w(t)} (J_1 + J_2)(k) \left| \mathcal{F}_s \right. \right\}.$$

From the fact that $|f_k e^{-i\lambda \Delta w(t)}| < 1$, and from Lemma 1, it is easily seen that

$$\lim_{n \to \infty} \sum_k E |f_k e^{-i\lambda \Delta w(t)} (J_1 + J_2)(k)| = 0.$$

Therefore, by letting $n \to \infty$ in (2.11), we have

$$E \left\{ e^{i\lambda \{v(t) - v(s)\}} - e^{i\lambda \{w(t) - w(s)\}} \left| \mathcal{F}_s \right. \right\} = 0.$$

Since $\mathcal{F}_s \subset \mathcal{F}_t$, Theorem 3 follows from the above identity and from assumption (1.2).

**Corollary 4.** The innovation process $\{v(t), \mathcal{F}_t\}$, $0 < t < T$, is a process of independent increments with the same probability law as $w(t)$.

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**Bibliography**


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