ON $\pi'$-CLOSURE OF $\pi$-HOMOGENEOUS GROUPS

PAMELA FERGUSON

Abstract. Let $\pi$ be a set of odd primes. It is known that $\pi'$-closed groups are $\pi$-homogeneous, but that the converse does not hold in general. In this paper we prove that a finite group $G$ which is $D_{\pi}$ and $\pi$-homogeneous is $\pi'$-closed.

Let $G$ be a finite group, and let $\pi(G)$ denote the set of prime divisors of $|G|$. Let $\pi$ be a subset of $\pi(G)$; then $G$ is $\pi$-homogeneous if $N_G(H)/C_G(H)$ is a $\pi$-group whenever $H$ is a nontrivial $\pi$-group. Let $\pi' = \pi(G) - \pi$; then $G$ is $\pi'$-closed if the set of $\pi'$-elements forms a subgroup of $G$.

It is known that $\pi'$-closed groups are $\pi$-homogeneous. The converse does not hold in general. For example, $A_5$ is $5'$-homogeneous, but $A_5$ is not 5-closed. Thus, some extra conditions are necessary in order that $\pi$-homogeneity implies $\pi'$-closure. We shall say that $G$ is a $D_{\pi}$ group if all maximal $\pi$-subgroups of $G$ are conjugate $S_{\pi}$-subgroups. It should be noted that we do not assume all $S_{\pi}$-subgroups of $G$ are solvable. Z. Arad [1] has conjectured that if $\pi$ is a set of primes, then $D_{\pi}$ and $\pi$-homogeneity imply $\pi'$-closure. This conjecture is proved in the theorem below.

Theorem. Let $G$ be a finite group such that $G$ is $D_{\pi}$ and $\pi$-homogeneous, then $G$ is $\pi'$-closed.

Throughout this paper we assume $G \in$ Hypothesis A.

Hypothesis A. Let $\pi$ be a set of primes. $G$ is a $D_{\pi}$ $\pi$-homogeneous group of minimal order such that $G$ is not $\pi'$-closed.

We will argue by contradiction to show that no such group $G$ ($G \in$ Hypothesis A) exists.

If $G \in$ Hypothesis A, clearly $\pi(G) \neq \pi$. Let $D$ denote a maximal $\pi$-subgroup of $G$. If $H$ is a subgroup of $G$, and $\tau$ is a set of primes, then let $|H|_{\tau}$ be defined by $|H| = |H|_{s, \tau}$ where $(s, \tau) = 1$.

The following two results were proved in [2].

Lemma 1 [2, Lemma 2.3]. Subgroups and epimorphic images of $\pi$-homogeneous groups are $\pi$-homogeneous.

Lemma 2 [2, Lemma 2.4]. If $K$ is a normal subgroup of the $\pi$-homogeneous group $H$, and if $K$ and $H/K$ are $\pi'$-closed, then $H$ is $\pi'$-closed.

Lemma 3. Assume $G \in$ Hypothesis A; then $|O_{\pi}(G)| = 1$.

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Proof. We will assume \( |O_\pi(G)| > 1 \) and obtain a contradiction.

Let \( \overline{H} \) denote the image of a subset \( H \subseteq G \) in \( G / O_\pi(G) \). Since \( G \) is \( D_\pi \), \( O_\pi(G) \subseteq D^\delta \) for all \( g \in G \). Order arguments imply that the subgroups \( D^\delta \) are \( S_\pi \) subgroups of \( \overline{G} \).

Let \( \overline{H} \) be a maximal \( \pi \) subgroup of \( \overline{G} \). The fundamental homomorphism theorem implies \( |H| \mid |G| \). Now \( G \) a \( D_\pi \) group implies \( H \subseteq D^\delta \) for \( g \in G \). Thus \( \overline{H} \subseteq D^\delta \). The maximality of \( H \) implies \( \overline{H} = D^\delta \). Hence \( \overline{G} = D_\pi \).

Lemma 1 implies \( \overline{G} \) is \( \pi \)-homogeneous. Now \( |O_\pi(G)| > 1 \) implies \( |G| < |G| \). Hence \( G \in \text{Hypothesis A} \) implies \( \overline{G} \) is \( \pi' \)-closed. Now \( O_\pi(G) \) is trivially \( \pi' \)-closed so that Lemma 2 implies \( G \) is \( \pi' \)-closed. Thus if \( G \in \text{Hypothesis A} \), then \( O_\pi(G) = 1 \).

Lemma 4. Assume \( G \in \text{Hypothesis A} \) and \( S \) is a nontrivial \( p \) subgroup of \( D \).

Then the following conditions hold:

(i) \( N_G(S) \) is \( \pi' \)-closed;
(ii) \( N_G(S) = N_D(S)O_\pi(C_G(S)) \);
(iii) \( S \subseteq D^w \) implies \( D^w = D^\pi \) where \( y \in O_\pi(C_G(S)) \).

Proof. Lemma 1 implies \( N_G(S) \) is \( \pi \)-homogeneous and Lemma 3 implies \( |N_G(S)| < |G| \). Hence, in order to show that \( N_G(S) \) is \( \pi' \)-closed it is sufficient to show that \( N_G(S) = D_e \).

Suppose \( K \) is a maximal \( \pi \) subgroup of \( N_G(S) \). Clearly \( K \supseteq S \). Since \( G \) is \( D_\pi \), \( K \subseteq D^\delta \) for some \( g \in G \). Hence \( K = N_D(S) \) where \( S \subseteq D^\delta \).

Let \( p^n = |D|^\delta \). We will proceed by induction on \( p^n / |S| \). If \( p^n / |S| = 1 \), then \( S \) is a Sylow \( p \) subgroup of \( D \). Suppose that \( S \subseteq D^w \cap D \). Sylow’s theorems imply \( S = S^w \) where \( d \in D \). Hence

\[
N_D^w(S) = N_D^w(S^w) = (N_D(S))^d^w.
\]

Now \( d^w \in N_G(S) \) implies

\[
(*) \quad N_D^w(S) = (N_D(S))^r \quad \text{where} \quad r \in N_G(S).
\]

Now let \( K \) be a maximal \( \pi \) subgroup of \( N_G(S) \); then \( K = N_D(S) \) where \( S \subseteq D^w \). Thus (*) implies \( K \) is conjugate in \( N_G(S) \) to \( N_D(S) \). In particular, \( N_D(S) \) is a maximal \( \pi \) subgroup of \( N_G(S) \). Moreover, all maximal \( \pi \) subgroups are conjugate in \( N_G(S) \) to \( N_D(S) \). It follows easily that \( N_G(S) \) is \( D_e \). Hence, \( N_G(S) \) is \( \pi' \)-closed. Since \( N_G(S) \) is \( \pi \)-homogeneous, we see that \( O_\pi(N_G(S)) = O_\pi(C_G(S)) \). Order arguments now imply \( N_G(S) = N_D(S)O_\pi(C_G(S)) \).

If \( S \subseteq D^w \), then \( S = S^d^w \) for \( d \in D \) implies \( d^w \in N_G(S) \). Thus, \( d^w = d_2 y \) where \( d_2 \in N_D(S) \) and \( y \in O_\pi(C_G(S)) \). Hence \( D^w = D^\pi \).

Now suppose \( p^n > p^n / |S| > 1 \) and the lemma is true for all \( T \) such that \( p^n / |T| < p^n / |S| \) (i.e., \( |T| > |S| \)).

Let \( S_1 \) be a Sylow \( p \) subgroup of \( N_G(S) \), and let \( K \) be a maximal \( \pi \) subgroup of \( N_G(S) \) such that \( S_1 \subseteq K \). Then \( K = N_D(S) \) where \( S \subseteq D^f \). Now \( S \) is not a Sylow \( p \) subgroup of \( D \). Hence Sylow’s theorems imply if \( T \) is a Sylow \( p \) subgroup of \( N_D(S) \), then \( |T| > |S| \). Moreover, \( T \subseteq S_1^r \) where \( r \in
Thus $T \subseteq K' \cap D \subseteq D^{r'} \cap D$. Since $|T| > |S|$, $D^{r'} = D^r$ where $y \in O_\pi(C_G(T))$. Hence, $D^r = D^{x^{-1}}$, which implies

$$N_D^r(S) = N_{D^{x^{-1}}}(S) = \left( N_D(S^{y^{-1}}) \right)^{x^{-1}}.$$ 

However, $S \subseteq T$ implies $y \in C_G(S)$. Hence, $N_D(S) = (N_D(S))^{y^{-1}}$ where $y^{x^{-1}} \in N_G(S)$. Thus $N_D(S)$ is conjugate to $N_D(S)$ in $N_G(S)$. In particular, $N_D(S)$ is a maximal $\pi$-subgroup of $N_G(S)$ and $T$ is a Sylow $p$-subgroup of $N_G(S)$.

Suppose $S \subseteq D^w$; then $S$ is not a Sylow $p$-subgroup of $D^w$. Hence if $V$ is a Sylow $p$-subgroup of $N_D(S)$, then $|V/S| > p$. Sylow's theorems imply $V \subseteq T'$ where $r \in N_G(S)$. Thus $V = T_1'$ where $T_1 \subseteq T$ and $T_1 \supset S$. Hence $T_1 = V^{x^{-1}} \subseteq D^{x^{-1}} \cap D$. Now $|T_1| > |S|$ implies $D^{x^{-1}} = D^x$ where $x \in O_\pi(C_G(T_1))$. Hence $x \in C_G(S)$, so that $D^x = D^y$, where $y = x^{-1} \in N_G(S)$. Thus

$$N_D^x(S) = N_D^y(S) = \left( N_D(S^{y^{-1}}) \right)^{x^{-1}} = \left( N_D(S) \right)^{x^{-1}}.$$ 

Thus $N_D^x(S)$ is conjugate in $N_G(S)$ to $N_D(S)$. In particular, $N_D^x(S)$ is a maximal $\pi$-subgroup of $N_G(S)$. If $K$ is any maximal $\pi$-subgroup of $N_G(S)$, then $K = N_D^x(S)$ for some $D^w \supseteq S$. Hence $K = (N_D(S))^{y^{-1}}$, where $z \in N_G(S)$. It follows that $N_D(S)$ is $D^w$. Lemmas 1 and 3 imply $N_G(S)$ is $\pi$-closed. Now $O_\pi(N_G(S)) = O_\pi(C_G(S))$, and order arguments imply $N_G(S) = N_D(S)O_\pi(C_G(S))$.

If $S \subseteq D^w$, then the previous paragraph implies $N_D^w(S)$ is a maximal $\pi$-subgroup of $N_G(S)$. Since $N_G(S) = N_D(S)O_\pi(C_G(S))$, $N_D^w(S) = (N_D(S))^{x^{-1}}$ for $x \in O_\pi(C_G(S))$. Thus $D^{w^{-1}} \cap D \supseteq T$. Now $|T| > |S|$ implies $D^{w^{-1}} = D^u$ where $u \in O_\pi(C_G(T))$. Thus $D^w = D^y$ where $y = ux$ clearly is contained in $O_\pi(C_G(S))$. The lemma follows by induction.

Lemma 5. Assume $G \in$ Hypothesis A. Let $z \in D^w$; then $C_G(z)$ is $\pi$-closed. Moreover, $z \in D^w$ implies $D^w = D^y$ where $y \in O_\pi(C_G(z))$.

Proof. Let $z$ be an element of minimal order such that the lemma fails. Suppose $|\langle z \rangle| = p^k$ where $p$ is a prime. Lemma 4 implies a contradiction. Thus we may assume $z = z_1z_2$ where $|\langle z_1 \rangle| = p^n > 1$. $|\langle z_2 \rangle|$, $p = 1$, and $z_2 \neq 1$. Clearly $C_G(z) = C_G(z_1) \cap C_G(z_2)$. Since $1 < |\langle z_i \rangle| < |\langle z \rangle|$ for $i = 1, 2$, $C_G(z_i)$ is $\pi$-closed for $i = 1, 2$. Hence $C_G(z)$ is $\pi$-closed.

Now $z \in D \cap D^w$ implies $z_i \in D \cap D^w$ for $i = 1, 2$. Lemma 4 implies $D^w = D^y$ where $y \in O_\pi(C_G(z_1))$. Hence, $z_2 \in D^y = D^w$ which implies $z_2^{-1} \in D$. Thus

$$[z_2, y^{-1}] = z_2^{-1}z_2^{y^{-1}} \in D \cap O_\pi(C_G(z_1)) = 1.$$ 

Therefore,

$$y \in O_\pi(C_G(z_1)) \cap O_\pi(C_G(z_2)) = O_\pi(C_G(z)).$$

Lemma 6. Assume $G \in$ Hypothesis A. Any two elements of $D^w$ which are conjugate in $G$ are also conjugate in $D$. 

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Proof. Suppose \( z_1 = z_2^w \) where \( z_1, z_2 \in D^\# \). Then \( z_1 \in D \cap D^w \). Lemma 5 implies \( D^w = D^y \) where \( y \in O_{\pi'}(C_G(z_1)) \). Hence \( wy^{-1} \in N_G(D) \). Since \( D \) is a Hall \( \pi \)-subgroup of \( G \), \(|N_G(D)/D| = |G|_{\pi'} \). Theorem 6.2.1 [5] implies \( N_G(D) = MD \) where \( M \) is a \( \pi' \)-subgroup of \( N_G(D) \). Since \( N_G(D) \) is \( \pi \)-homogeneous we see that \( M \subseteq C_G(D) \). Hence, \( N_G(D) = M \times D \). Now \( wy^{-1} \in N_G(D) \) implies \( w = duy^{-1} \) where \( u \in M \). Thus \( z_1 = z_2^w \) implies \( z_2 = z_1^{y^{-1}u^{-1}d^{-1}} = z_1^{d^{-1}} \). Thus \( z_1 \) is conjugate to \( z_2 \) in \( D \).

Proof of Theorem. In order to prove the Theorem, we will show that \( G \) satisfies the hypothesis of Theorem 8.22 of [6] with \( H = D \). Let \( E \) be an elementary subgroup of \( G \) such that \(|E| \mid |D| \). Since \( G \) is \( D^g, E \subseteq D^\# \) for some \( g \in G \). Hence \( E \) is conjugate to a subgroup of \( D \). Lemma 6 implies whenever two elements of \( D \) are conjugate in \( G \), then they are conjugate in \( D \). Theorem 8.22 [6] now implies \( G \) is \( \pi' \)-closed.

References

Department of Mathematics, University of Miami, Coral Gables, Florida 33124