A NONSPECTRAL BIRKHOFF-REGULAR DIFFERENTIAL OPERATOR

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ABSTRACT. It is shown by example that Birkhoff-regular differential operators need not be spectral.

In his well-known paper [2, p. 222] Dunford stated that Birkhoff-regular operators are spectral. While many are spectral [3, XIX.4.16] we have found that some are not.

Our example is the operator L induced in $L^2(0,1)$ by the boundary value problem

\begin{align}
(1) \quad &-y'' = \lambda y \quad \text{on } [0,1], \\
(2) \quad &y'(0) - y'(1) - y(1) = 0, \quad \text{and} \quad y(0) = 0.
\end{align}

$L$ is given by $Ly = -y''$ for all $y$ in $L^2(0,1)$ such that $y'$ is absolutely continuous on $[0,1]$, (2) holds and $y'' \in L^2(0,1)$.

It may be verified that (1)–(2) is regular in the sense of Birkhoff ([5] or [4, §4.8]), and following [1, §§7.2 and 12.2] or [4, §3.7] it may be verified that, for each positive integer $k$, $(2k\pi)^2$ is an eigenvalue. (There are other eigenvalues but these are not important for our purpose here.)

By computing the Green's function and calculating its residue at $(2k\pi)^2$ [4, §3.8], it may be concluded that, for each positive integer $k$, the projection $E((2k\pi)^2; L)$, which we abbreviate by $E_k$, is given by

\[
(E_kf)(t) = -\int_0^1 (\sin 2k\pi t)(4k\pi \cos 2k\pi s - 2\sin 2k\pi s)f(s)ds
\]

for each $f \in L^2(0,1)$.

Let $f_k(s) = \cos 2k\pi s$ for each positive integer $k$ and each $s \in [0,1]$. Then $\|f_k\| = \sqrt{2}/2$ and $\|E_kf_k\| = \sqrt{2}k\pi$ for each positive integer $k$ ($\|\cdot\|$ is the usual $L^2(0,1)$ norm). Thus the sequence of projections \{\E_k\} is not bounded, so by [3, XVIII.2.33], $L$ is not spectral.

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188

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REFERENCES


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