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OPERATORS IN THE COMMUTANT OF A REDUCTIVE ALGEBRA

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ABSTRACT. Let $\mathcal{A}$ be a reductive algebra. It is shown that there is a subspace $\mathfrak{M}$ that reduces $\mathcal{A}$ and such that the commutant of $\mathcal{A}|\mathfrak{M}$ is selfadjoint and the commutant of $\mathcal{A}|\mathfrak{M}^{\perp}$ consists of hyporeductive operators. It is then shown that under a variety of conditions, if an operator T is in $\mathcal{A}'$, then T^* is in $\mathcal{A}'$.

In [5], we made use of a decomposition of reductive operators on Hilbert space to deduce some results concerning the selfadjointness of the commutants of such operators. In that paper we also observed that a similar decomposition of a reductive operator algebra is possible. In this paper we produce such a decomposition and use it to answer a number of questions first raised by Rosenthal [9].

By an algebra, we will mean a weakly closed subalgebra (with identity) of the algebra of all (bounded) operators on a separable Hilbert space. If $\mathcal{A}$ is an algebra and $\mathfrak{M}$ a subspace of $\mathcal{H}$ then $\mathcal{A}\mathfrak{M} = \{Af: A \in \mathcal{A} \text{ and } f \in \mathfrak{M}\}$, and $\mathfrak{M}$ is *invariant* for $\mathcal{A}$ if $\mathcal{A}\mathfrak{M} \subseteq \mathfrak{M}$. The lattice of subspaces invariant for $\mathcal{A}$ is denoted by Lat $\mathcal{A}$. We also denote by $\mathcal{A}^*$ the algebra $\{A^*: A \in \mathcal{A}\}$, and by $\mathcal{A}'$ the algebra $\{B: AB = BA \text{ for all } A \in \mathcal{A}\}$. Finally, $\mathcal{A}$ is *reductive* if $\mathfrak{M} \in \operatorname{Lat} \mathcal{A}$ implies $\mathfrak{M}^{\perp} \in \operatorname{Lat} \mathcal{A}$, or equivalently, if $\operatorname{Lat} \mathcal{A} = \operatorname{Lat} \mathcal{A}^*$.

The results for which we are aiming are Theorems 2 and 3, which state that for every reductive algebra $\mathcal{A}$ there is a subspace $\mathfrak{M}_0 \in \operatorname{Lat} \mathcal{A} \cap \operatorname{Lat} \mathcal{A}'$, such that $\mathcal{A}'|\mathfrak{M}_0$ is selfadjoint and $\operatorname{Lat}(\mathcal{A}|\mathfrak{M}_0^{\perp}) \subseteq \operatorname{Lat}(\mathcal{A}'|\mathfrak{M}_0^{\perp})$. Before we begin, we remark that certain techniques developed by Hoover [4] will also yield Theorems 2 and 3.

THEOREM 1. *Let $\mathcal{A}$ be a reductive algebra and $\mathfrak{M} \in \operatorname{Lat} \mathcal{A}$. Let $X: \mathfrak{M}^{\perp} \to \mathfrak{M}$ and suppose that $T \in \mathcal{A}'$, where T has the form*

$$T = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$$

according to the decomposition $\mathcal{H} = \mathfrak{M} \oplus \mathfrak{M}^{\perp}$. Then

(1) $T^* \in \mathcal{A}'$.

(2) $(\operatorname{ran} T)^- \in \operatorname{Lat} \mathcal{A}$ *and* $(\mathcal{A}|(\operatorname{ran} T)^-)'$ *is selfadjoint.*

(3) $\ker T \in \operatorname{Lat} \mathcal{A}$ *and* $(\mathcal{A}|\ker^{\perp} T)'$ *is selfadjoint.*

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PROOF. (1) All decompositions of vectors and operators are with respect to $\mathfrak{M}$ and $\mathfrak{M}^\perp$. Let $\mathfrak{N}$ be the subspace $\{\langle Xf, f\rangle : f \in \mathfrak{M}^\perp\}$. Since $T \in \mathcal{Q}'$ it is easy to check that $\mathfrak{N} \in \operatorname{Lat} \mathcal{Q}$, and thus the subspace $\mathfrak{N}^\perp = \{\langle g, -X^*g\rangle : g \in \mathfrak{M}\}$ also lies in $\operatorname{Lat} \mathcal{Q}$.

If $A \in \mathcal{Q}$ we can decompose A as $A_1 \oplus A_2$, since $\mathcal{Q}$ is reductive. If $g \in \mathfrak{M}$ we have

$$(A_1 \oplus A_2)\langle g, -X^*g\rangle = \langle A_1 g, -A_2X^*g\rangle.$$

Since this vector must lie in $\mathfrak{N}^\perp$ it follows that $-X^*A_1g = -A_2X^*g$ for any $g \in \mathfrak{M}$, and hence that $X^*A_1 = A_2X^*$, which implies that $T^* \in \mathcal{Q}'$.

(2) Since $T \in \mathcal{Q}'$, $(\operatorname{ran} T)^- \in \operatorname{Lat} \mathcal{Q}$. Let C be an operator on $(\operatorname{ran} T)^-$ that lies in the commutant of $\mathcal{Q}|(\operatorname{ran} T)^-$; we want to show that $C^* \in (\mathcal{Q}|(\operatorname{ran} T)^-)'$. Let C_1 be the operator on $\mathcal{H}$ defined by

$$C_1 f = \begin{cases} Cf, & f \in (\operatorname{ran} T)^-, \\ 0, & f \in \operatorname{ran}^\perp T. \end{cases}$$

Then $C_1 \in \mathcal{Q}'$ and it will suffice to show that $C_1^* \in \mathcal{Q}'$.

Let $T_1 = C_1T$; then $T_1 \in \mathcal{Q}'$. Moreover, since $\mathfrak{M}$ is invariant under C_1, it is easy to see that T_1 has the same form as T, namely

$$T_1 = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}.$$

It follows by part (1) that $T_1^* \in \mathcal{Q}'$, that is, for any $A \in \mathcal{Q}$, $T^*C_1^*A = AT^*C_1^*$. Since $T^* \in \mathcal{Q}'$, $T^*(AC_1^* - C_1^*A) = 0$, that is, $\operatorname{ran}(AC_1^* - C_1^*A) \subseteq \ker T^*$. On the other hand $\operatorname{ran} C_1^* \subseteq (\operatorname{ran} T)^-$ and $(\operatorname{ran} T)^-$ reduces A, so that

$$\operatorname{ran}(AC_1^* - C_1^*A) \subseteq (\operatorname{ran} T)^- = \ker^\perp T^*.$$

We conclude that $AC_1^* - C_1^*A = 0$, and $C_1^* \in \mathcal{Q}'$.

(3) Let U be the unitary operator $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Since $T^* \in \mathcal{Q}'$, we have

$$\begin{pmatrix} 0 & X^* \\ 0 & 0 \end{pmatrix} = U^*T^*U \in (U^*\mathcal{Q}U)'.$$

By part (2), $[(U^*\mathcal{Q}U)|(\operatorname{ran} U^*T^*U)^-]'$ is selfadjoint. Thus $[\mathcal{Q}|(\operatorname{ran} T^*)^-]'$ is selfadjoint and (3) follows because $(\operatorname{ran} T^*)^- = \ker^\perp T$.

COROLLARY. *Let $\mathcal{Q}$ be reductive and $\mathfrak{M} \in \operatorname{Lat} \mathcal{Q}$. Let $Y: \mathfrak{M} \to \mathfrak{M}^\perp$ and suppose that $S \in \mathcal{Q}'$, where $S = \left(\begin{smallmatrix} 0 & 0 \\ Y & 0 \end{smallmatrix}\right)$. Then*

(1) $S^* \in \mathcal{Q}'$.

(2) $\ker S \in \operatorname{Lat} \mathcal{Q}$ *and* $(\mathcal{Q}|\ker^\perp S)'$ *is selfadjoint.*

(3) $(\operatorname{ran} S)^- \in \operatorname{Lat} \mathcal{Q}$ *and* $(\mathcal{Q}|\operatorname{ran} S)'$ *is selfadjoint.*

PROOF. Consider adjoints and apply the theorem.

THEOREM 2. *Let $\mathcal{Q}$ be a reductive algebra. There is a subspace $\mathfrak{M}_0$ such that*

(1) $\mathfrak{M}_0 \in \operatorname{Lat} \mathcal{Q}$;

(2) $(\mathcal{Q}|\mathfrak{M}_0)'$ *is selfadjoint*;

(3) *there is no nonzero subspace $\mathfrak{M} \subseteq \mathfrak{M}_0^\perp$ with properties* (1) *and* (2).

Moreover, this subspace $\mathfrak{M}_0$ reduces $\mathfrak{A}'$ as well.

PROOF. Let $\mathfrak{F} = \{\mathfrak{M} \in \operatorname{Lat} \mathfrak{A} : (\mathfrak{A}|\mathfrak{M})' \text{ is selfadjoint}\}$. The family $\mathfrak{F}$ is nonempty since it contains the zero subspace. Suppose that $\{\mathfrak{M}_\alpha\}_{\alpha \in B}$ is a chain in $\mathfrak{F}$; in order to apply Zorn's lemma we would like to show that $\mathfrak{M} = \bigvee\{\mathfrak{M}_\alpha : \alpha \in B\}$ is also in $\mathfrak{F}$, for which it suffices to show that $(\mathfrak{A}|\mathfrak{M})'$ is selfadjoint.

Let T be an operator on $\mathfrak{M}$ such that $T \in (\mathfrak{A}|\mathfrak{M})'$. To show that $T^* \in (\mathfrak{A}|\mathfrak{M})'$ we must show that for all $A \in \mathfrak{A}|\mathfrak{M}$ and for all $f \in \mathfrak{M}$ we have $T^*Af = AT^*f$, and in fact it will be enough to show this equality for all $f \in \bigcup\{\mathfrak{M}_\alpha : \alpha \in B\}$, because this set is dense in $\mathfrak{M}$. On the other hand, in this case $f \in \mathfrak{M}_\beta$ for some $\beta \in B$.

Decompose $\mathfrak{M}$ as $\mathfrak{M}_\beta \oplus (\mathfrak{M} \ominus \mathfrak{M}_\beta)$; then (since $\mathfrak{M}_\beta \in \operatorname{Lat} \mathfrak{A}$) if $A \in \mathfrak{A}|\mathfrak{M}$ we have

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}.$$

Because $f \in \mathfrak{M}_\beta$ we have

$$T^*Af = \langle T_{11}^* A_1 f, T_{12}^* A_1 f \rangle \quad \text{and} \quad AT^*f = \langle A_1 T_{11}^* f, A_2 T_{12}^* f \rangle.$$

Since $T_{11} \in (\mathfrak{A}|\mathfrak{M}_\beta)'$ and $(\mathfrak{A}|\mathfrak{M}_\beta)'$ is selfadjoint, we have $T_{11}^* A_1 = A_1 T_{11}^*$. Furthermore, the operator $\left(\begin{smallmatrix} 0 & T_{12} \\ 0 & 0 \end{smallmatrix}\right)$ lies in $(\mathfrak{A}|\mathfrak{M})'$ and by Theorem 1, so does its adjoint $\left(\begin{smallmatrix} 0 & 0 \\ T_{12}^* & 0 \end{smallmatrix}\right)$ and it follows that $A_2 T_{12}^* = T_{12}^* A_1$. Thus $T^*Af = AT^*f$, $T^* \in (\mathfrak{A}|\mathfrak{M})'$, and $\mathfrak{M} \in \mathfrak{F}$.

By Zorn's lemma there exists a maximal element $\mathfrak{M}_0$ of $\mathfrak{F}$. $\mathfrak{M}_0$ automatically satisfies requirements (1) and (2) of the theorem. Suppose there is a nonzero subspace $\mathfrak{M}_1$ of $\mathfrak{M}_0^\perp$ for which (1) and (2) hold. We assert that $\mathfrak{M}_0 \oplus \mathfrak{M}_1$ lies in $\mathfrak{F}$, a fact which contradicts the maximality of $\mathfrak{M}_0$. We must show that $(\mathfrak{A}|\mathfrak{M}_0 \oplus \mathfrak{M}_1)'$ is selfadjoint. If S is an operator on $\mathfrak{M}_0 \oplus \mathfrak{M}_1$ such that $S \in (\mathfrak{A}|\mathfrak{M}_0 \oplus \mathfrak{M}_1)'$ then we decompose S as

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$

where $S_{11} \in (\mathfrak{A}|\mathfrak{M}_0)'$ and $S_{22} \in (\mathfrak{A}|\mathfrak{M}_1)'$. By assumption $S_{11}^* \in (\mathfrak{A}|\mathfrak{M}_0)'$ and $S_{22}^* \in (\mathfrak{A}|\mathfrak{M}_1)'$. Moreover, the operator

$$\begin{pmatrix} 0 & S_{12} \\ 0 & 0 \end{pmatrix}$$

satisfies the hypotheses of Theorem 1 for the reductive algebra $(\mathfrak{A}|\mathfrak{M}_0 \oplus \mathfrak{M}_1)$. Thus by that theorem the operator

$$\begin{pmatrix} 0 & 0 \\ S_{12}^* & 0 \end{pmatrix}$$

lies in $(\mathfrak{A}|\mathfrak{M}_0 \oplus \mathfrak{M}_1)'$. Similarly we use the corollary to Theorem 1 to show that $\left(\begin{smallmatrix} 0 & S_{21}^* \\ 0 & 0 \end{smallmatrix}\right)$ lies in $(\mathfrak{A}|\mathfrak{M}_0 \oplus \mathfrak{M}_1)'$. Since

$$S^* = \begin{pmatrix} S_{11}^* & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & S_{22}^* \end{pmatrix} + \begin{pmatrix} 0 & S_{21}^* \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ S_{12}^* & 0 \end{pmatrix}$$

we see that $S^* \in (\mathcal{A}|\mathfrak{M}_0 \oplus \mathfrak{M}_1)'$, and thus that $(\mathcal{A}|\mathfrak{M}_0 \oplus \mathfrak{M}_1)'$ is selfadjoint.

Finally, to show that $\mathfrak{M}_0 \in \operatorname{Lat} \mathcal{A}'$, suppose that $T \in \mathcal{A}'$ and write

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

according to the decomposition $\mathcal{H} = \mathfrak{M}_0 \oplus \mathfrak{M}_0^\perp$.

Since $\mathfrak{M}_0$ reduces $\mathcal{A}$ the operator

$$S = \begin{pmatrix} 0 & T_{12} \\ 0 & 0 \end{pmatrix}$$

also lies in $\mathcal{A}'$, and by Theorem 1, ker S reduces $\mathcal{A}$ and $(\mathcal{A}|\ker^\perp S)'$ is selfadjoint. However, $\ker^\perp S$ is a subspace of $\mathfrak{M}_0^\perp$ so by the maximality of $\mathfrak{M}_0$ it must be that $\ker^\perp S = \{0\}$, that is, $T_{12} = 0$. Similarly we can show that $T_{21} = 0$ and thus $\mathfrak{M}_0$ reduces T.

We use the notation Red $\mathfrak{B}$ to mean Lat $\mathfrak{B} \cap$ Lat $\mathfrak{B}^*$, where $\mathfrak{B}$ is any algebra.

THEOREM 3. *Let $\mathcal{A}$ be reductive and suppose that for no nonzero subspace $\mathfrak{M}$ in* Lat $\mathcal{A}$ *is it true that $(\mathcal{A}|\mathfrak{M})'$ is selfadjoint. Then* Lat $\mathcal{A} \subseteq$ Red $\mathcal{A}'$.

PROOF. Suppose $\mathfrak{M} \in$ Lat $\mathcal{A}$ and $T \in \mathcal{A}'$. Decompose $\mathcal{H}$ as $\mathfrak{M} \oplus \mathfrak{M}^\perp$ and T as

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix};$$

let

$$S = \begin{pmatrix} 0 & T_{12} \\ 0 & 0 \end{pmatrix}.$$

By Theorem 1, $(\operatorname{ran} S)^- \in$ Lat $\mathcal{A}$ and $[\mathcal{A}|(\operatorname{ran} S)^-]'$ is selfadjoint. Hence, by the hypothesis of the theorem, ran $S = \{0\}$, that is, $T_{12} = 0$. Similarly $T_{21} = 0$ and $\mathfrak{M} \in$ Red $\mathcal{A}'$.

C. K. Fong [1] has used the word *hyporeductive* to refer to an operator T such that every hyperinvariant subspace of T reduces T.

COROLLARY. *Let $\mathcal{A}$ be as in Theorem* 3. *If $T \in \mathcal{A}'$ and $\mathfrak{M}$ is hyperinvariant for T then $\mathfrak{M} \in$* Red $\mathcal{A}'$. *In particular, T is hyporeductive.*

It follows from all the above that if $\mathcal{A}$ is any reductive algebra and $T \in \mathcal{A}'$, then $T = T_1 \oplus T_2$ where $T_1^* \in (\mathcal{A}|\mathfrak{M}_0)'$ and T_2 is hyporeductive. Thus if we desire to show that $T^* \in \mathcal{A}'$, it suffices to show that $T_2^* \in (\mathcal{A}|\mathfrak{M}_0^\perp)'$.

In [9], P. Rosenthal introduced the following property which an operator T may have in connection with reductive algebras:

(P) If $\mathcal{A}$ is any reductive algebra such that $\mathcal{A}'$ contains T, then $\mathcal{A}'$ contains T^*.

Rosenthal then asked if T has property (P) under each of the following conditions:

(1) T is polynomially compact,

(2) $1 - T^*T$ is in some C_p class,

(3) $T^* - T$ is in some C_p class,

(4) T is a part of a finite-multiplicity backward shift.

We will show that each of the above conditions implies (P), but we need a preliminary result (Lemma 2).

In [5], the following lemma is proved:

LEMMA 1. *Let C be a nonzero compact operator, and suppose that B is an operator such that every subspace that reduces both B and C and has dimension greater than* 1 *properly contains a nonzero subspace that reduces B and C. Then B and C have a common reducing eigenvector.*

The argument used to prove this lemma, with minor (and obvious) modifications will yield the following fact:

LEMMA 2. *Let B be an operator and C a nonzero compact operator. Suppose that*

(1) *Every hyperinvariant subspace of B reduces B and C.*

(2) *Every hyperinvariant subspace of B of dimension greater than* 1 *properly contains a nonzero hyperinvariant subspace of B.*

Then B and C have a common reducing eigenvector.

PROOF. See [5, p. 230].

We are now ready to answer Rosenthal's questions. It should be remarked that C. K. Fong [2] has proved part (1) of the following theorem.

THEOREM 4. *An operator T has property* (P) *under any one of the following conditions*:

(1) *T is polynomially compact,*

(2) *$T^* - T$ is in some C_p class,*

(3) *$1 - T^*T$ is in some C_p class,*

(4) *T is a part of some finite-multiplicity backward shift.*

PROOF. (4) follows from (3), because the multiplicity of the shift of which T is a part is the rank of $\sqrt{1 - T^*T}$. (See [3, p. 278].) If this rank is finite then so is the rank of $1 - T^*T$.

Next we remark that each of the conditions (1), (2), (3) is inherited by direct summands, and that each condition guarantees the existence of hyperinvariant subspaces [8, Corollaries 6.13, 6.15, 6.16] and [7, Theorem 1.1].

We will prove in detail that (2) implies that (P) holds; the proofs for (1) and (3) are analogous. Let $C = T^* - T$ and suppose that C is in some C_p class. We also suppose that $\mathcal{A}$ is a reductive algebra and that $T \in \mathcal{A}'$. Let $\mathfrak{M}_0$ be

the subspace of Theorem 2; note that $\mathfrak{M}_0$ reduces $\mathcal{A}$, T, and C, and that $(T^*|\mathfrak{M}_0) \in (\mathcal{A}|\mathfrak{M}_0)'$. Thus it suffices to consider the case where T is hyporeductive (by the remark following the Corollary to Theorem 3).

Since T is hyporeductive, the space $\mathfrak{M}_1$ spanned by all the eigenvectors of T reduces $\mathcal{A}$, T, and C, and $(T^*|\mathfrak{M}_1) \in (\mathcal{A}|\mathfrak{M}_1)'$ (the last statement follows by Lemma 5 of [**1**]); thus it suffices to consider the restriction of $\mathcal{A}$, T, and C to $\mathfrak{M}_1^{\perp}$; i.e., we consider the case where T has no eigenvalues and is hyporeductive.

After these reductions suppose C is nonzero. Because T is hyporeductive, every hyperinvariant subspace of T reduces T and C. Further, suppose $\mathfrak{M}$ is a hyperinvariant subspace of T, of dimension greater than 1. Then $\mathfrak{M}$ reduces T and C, and $(T|\mathfrak{M})^* - (T|\mathfrak{M})$ lies in some C_p class. Thus there is a hyperinvariant subspace of T properly contained in $\mathfrak{M}$. It now follows from the assumption that C is nonzero, and from Lemma 2, that T has a reducing eigenvector; however, we reduced to the case where T has no eigenvectors. Thus it must be that $C = 0$, which means that $T^* = T$ and $T^* \in \mathcal{A}'$. The proof is complete.

To show that (1) implies (P) let p be a polynomial such that $p(T) = C$ is compact and proceed as above. It is necessary to know that an algebraic hyporeductive operator is normal [**1**, Theorem 4].

To show that (3) implies (P) let $C = 1 - T^*T$ and proceed as above.

We remark that the proof of Theorem 4 also establishes the following fact :

COROLLARY. *If T is hyporeductive and any one of conditions* (1) *through* (4) *holds, then T is normal.*

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