A SUPERPOSITION THEOREM FOR BOUNDED CONTINUOUS FUNCTIONS

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Abstract. It is shown that there exist \(2n + 1\) real valued, continuous functions \(\phi_0, \ldots, \phi_{2n}\) defined on \(\mathbb{R}^n\) such that every bounded real valued continuous function on \(\mathbb{R}^n\) is expressible in the form \(\sum_{i=0}^{2n} g_i \phi_i\) for some \(g \in C(\mathbb{R})\). Extensions to some unbounded functions are also made.

The Kolmogorov superposition theorem states, in the form of Lorentz [4], that there exist \(2n + 1\) real valued functions \(\phi_1, \ldots, \phi_{2n+1}\) continuous on the closed unit cube, \(I^n\), in \(\mathbb{R}^n\) such that for any \(f \in C(I^n)\) there is \(g \in C(\mathbb{R})\) such that

\[
f(x) = \sum_{i=1}^{2n+1} g\left(\phi_i(x)\right) \quad \text{for all } x \in I^n.
\]

In fact one may take \(\phi_i\) to be of the form \(\phi_i(x_1, \ldots, x_n) = \sum_{p=1}^{n} e^{\psi_i(x_p)}\) where \(\psi_i \in C(I)\). The various proofs of this result, e.g., [2], [3], [4], all make use of the uniform continuity of \(f\). Recently, Doss [1], has proven the following theorem.

Theorem. There are functions \(\psi_1, \ldots, \psi_{2n-1}, \phi_1, \ldots, \phi_{2n+1} \in C(\mathbb{R}^n)\) such that for any \(f \in C(\mathbb{R}^n)\) there exist \(g, h \in C(\mathbb{R})\) such that

\[
f(x) = \sum_{i=1}^{2n-1} h\left(\psi_i(x)\right) + \sum_{i=1}^{2n+1} g\left(\phi_i(x)\right) \quad \text{for all } x \in \mathbb{R}^n.
\]

Our main result is that there is a choice of \(\phi_1, \ldots, \phi_{2n+1}\) such that for bounded functions (and some unbounded ones) we may take \(h \equiv 0\). It has as an immediate corollary that every \(f \in C(\mathbb{R}^n)\) can be represented in terms of \(\phi_1, \ldots, \phi_{2n+1}\), the tangent function, and a function of a single variable. In the spirit of Hedberg [2], we obtain the existence of the \(\phi_i\)'s by means of a category argument. We then show that if \(f\) has compact support, then the representing function \(g\) must decay exponentially to zero away from the support of \(f\). Paracompactness of \(\mathbb{R}^n\) enables us to complete the proof. The norm used on functions is always the uniform norm on \(\mathbb{R}^n\) unless stated otherwise.

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Throughout this paper \( n > 2 \) is a fixed but arbitrary integer and for \( m > 0 \) we define the following subsets of \( \mathbb{R}^n \): \( B_m = \{ x : \| x \| < m \} \), \( S_m = \{ x : \| x \| = m \} \), and \( R_m = \{ x : m - 1 < \| x \| < m \} \); here we use the \( l_\infty \) norm. We now recall the cubes of Kolmogorov, [3], [4].

**Lemma 1.** There exist closed, bounded cubes \( \{ c_{j,k}^q : 1 < q < 2n + 1, k > 1, i \in \mathbb{N} \} \) and strictly decreasing null sequences \( \{ a_k \}_{k \geq 1}, \{ b_k \}_{k \geq 1} \) such that

1. For fixed \( k \) each point of \( \mathbb{R}^n \) is contained in at least \( n + 1 \) of the cubes \( \{ c_{j,k}^q : 1 < q < 2n + 1, i \in \mathbb{N} \} \).
2. For fixed \( k \) and \( q \), the cubes \( \{ c_{j,k}^q : i \in \mathbb{N} \} \) are disjoint.
3. \( \text{diam}(c_{j,k}^q) < a_k \) for all \( i, q \).
4. \( \min_i H(c_{j,k}^q, c_{j,k}^q) < b_k \) for all \( j, q \) where \( H(A, B) \) denotes the Hausdorff distance between \( A \) and \( B \).

The next result shows that the families of cubes can be separated by functions with prescribed ranges.

**Lemma 2.** There are \( 2n + 1 \) functions \( \phi_1, \ldots, \phi_{2n+1} \) in \( C(\mathbb{R}^n) \) such that

1. for \( 1 < q < 2n + 1 \) and \( m > 1 \), \( \phi_q(R_m) = [m, m + 2] \) and \( \phi_q(S_m) = [m + 1, m + 3] \).
2. for each \( N > 1 \), there is a \( k > N \) such that the sets \( \phi_q(c_{j,k}^q), 1 < q < 2n + 1, i \in \mathbb{N} \), are mutually disjoint.

**Proof.** The set \( M = \{ (\psi_1, \ldots, \psi_{2n+1}) \in [C(\mathbb{R}^n)]^{2n+1} : \text{each } \psi_q \text{ satisfies (a)} \} \) is a nonempty complete metric space if it is given the product-compact-open topology. For \( k > 1 \) let \( \emptyset_k = \{ (\psi_q) \in M : \text{the sets } \psi_q(c_{j,k}^q), 1 < q < 2n + 1, i \in \mathbb{N} \text{ are mutually disjoint} \} \). It can be verified that for each \( N > 1 \), \( \bigcup_{k \geq N} \emptyset_k \) is open and dense in \( M \). By the Baire category theorem \( \bigcap_{N > 1} \bigcup_{k \geq N} \emptyset_k \) is nonvoid; this is statement (b). Q.E.D.

**Lemma 3.** Let \( f \in C(\mathbb{R}^n) \) with \( \text{supp } f \subseteq \text{int}(\bigcup_{j=0}^l R_{m+j}) \) and let \( n(n + 1)^{-1} < \theta < 1 \). There exists \( g \in C(\mathbb{R}) \) such that

1. \( \| g \| < (n + 1)^{-1} \| f \| \),
2. \( |f(x) - \sum_q g(\phi_q(x))| < \theta \| f \| \), for all \( x \in \mathbb{R}^n \) and
3. \( \text{supp } g \subseteq [m - 1, m + l + 3] \).

**Proof.** Following Lorentz [4, p. 173], we let \( \varepsilon > 0 \) be such that \( n(n + 1)^{-1} + \varepsilon < \theta \). Let \( N \) be so large that if \( \| x - y \| < a_N \) (cf. Lemma 1), then \( |f(x) - f(y)| < \varepsilon \| f \| \). Let \( k > N \) be such that the sets \( \phi_q(c_{j,k}^q), 1 < q < 2n + 1, i \in \mathbb{N} \), are mutually disjoint. We define \( g \) to be constant on each set \( \phi_q(c_{j,k}^q) \) the constant being the value of \( (n + 1)^{-1} x \) at the center of \( c_{j,k}^q \). Extending \( g \) linearly to all of \( \mathbb{R} \) we see that (a) is satisfied. The proof of (b) follows as in [4]. To prove (c) note that \( g(\phi_q(c_{j,k}^q)) \neq \{0 \} \) only if \( c_{j,k}^q \cap (\bigcup_{j=0}^l R_{m+j}) = \emptyset \) and use part (a) of Lemma 2.

The next result shows that if \( f(x) \) has compact support, then it can be represented in the form \( (*) \) where the function \( g(x) \) decays to zero exponentially as \( x \to \infty \).
Lemma 4. Let $f \in C(\mathbb{R}^n)$ with $\text{supp } f \subseteq \text{int}(R_m \cup R_{m+1})$ and let $\theta$ be as in Lemma 3. There exists $g \in C(\mathbb{R})$ and a constant $C$ depending on only $\theta$ such that

(a) for all $x \in \mathbb{R}^n$, $f(x) = \sum_{q=0}^{\infty} g(\phi_q(x))$ and

(b) $\|g\|_{L^\infty([k,k+1])} < C\|f\|\theta^{m-k}$ for all $k > 0$.

Proof. Let $g_1$ satisfy the conditions of Lemma 3 and define $f_1(x) = f(x) - \sum_{i} g_1(\phi_i(x))$. Note that $\text{supp } f_1 \subseteq \bigcup_{j=1}^{3} R_{m+j}$. In general, if $f_k$ is defined, we let $g_{k+1}$ be such that $\|g_{k+1}\| < (n+1)^{-1}\|f_k\|$ and $|f_k(x) - \sum_{i} g_{k+1}(\phi_i(x))| < \theta\|f_k\|$ and define $f_{k+1}(x) = f_k(x) - \sum_{i} g_{k+1}(\phi_i(x))$. As in [4], we have $\|f_k\| < \theta^k\|f\|$ and $\|g_{k+1}\| < \theta^k\|f\|$ which yield (a). By Lemma 3 the $g_k$'s can be chosen so that $\text{supp } f_k \subseteq \bigcup_{j=3k+1}^{3k+4} R_{m+j}$ and $\text{supp } g_{k+1} \subseteq [m - 3k - 1, m + 3k + 4] \cap [0, \infty)$. Consequently, on any interval of the form $[0, m - 3k - 1]$ or $[m + 3k + 4, \infty)$ the norm of $g$ is bounded by $\|f\|\Sigma_{i=0}^{n+k+1}\theta^i$. This implies (b). Q.E.D.

Theorem 1. Let $f \in C(\mathbb{R}^n)$ be bounded. There exists $g \in C(\mathbb{R})$ such that for all $x \in \mathbb{R}^n$, $f(x) = \sum_{m=0}^{\infty} f_m(x)$ where $\text{supp } f_m \subseteq \text{int}(R_m \cup R_{m+1})$ and $\|f_m\| < \|f\|_{L^\infty([m,m+1])}$. Let $g_m$ satisfy the conclusions of Lemma 4 for $f_m$. So,

$$f(x) = \sum_{m=0}^{\infty} \sum_{i=1}^{2n+1} g_m(\phi_i(x)),$$

and

$$\|g_m\|_{L^\infty([k,k+1])} < C\theta^{m-k}\|f\|_{L^\infty([m,m+1])}.$$ 

By the Weierstrass $M$-test, $\sum_{m=0}^{\infty} g_m \in C[k, k+2]$ for all $k > 0$. Therefore, $\sum_{m=0}^{\infty} g_m \in C(\mathbb{R})$ and $f(x) = \sum_{i=1}^{2n+1} (\sum_{m=0}^{\infty} g_m(\phi_i(x)))$. Q.E.D.

Corollary 1. Let $f \in C(\mathbb{R}^n)$ and assume that $\sum_{m=0}^{\infty} \theta^m\|f\|_{L^\infty([m,m+1])}$ converges for some $n(n+1)^{-1} < \theta < 1$, then there is $g \in C(\mathbb{R})$ such that

$$f = \sum_{i=1}^{2n+1} g \circ \phi_i.$$ 

Proof. The assumptions insure that the series $\sum_{m=0}^{\infty} g_m$ converges uniformly on compact subsets of $\mathbb{R}$.

We have not been able to extend Theorem 1 to all unbounded functions. However, it is possible to represent unbounded functions by nonlinear superpositions of the $\phi_i$'s.

Corollary 2. There is a constant $K > 0$ such that for $f \in C(\mathbb{R}^n)$, there exists $g \in C(\mathbb{R})$ with $\|g\| < K$ and $f(x) = \tan\{\sum_{i=1}^{2n+1} g(\phi_i(x))\}$ for all $x \in \mathbb{R}^n$.

Proof. Since $f(x) = \tan\{\text{Arc tan } f(x)\}$, apply Theorem 1 to $\text{Arc tan } f(x)$.

Remark. Theorem 1 can be extended to more general spaces than $\mathbb{R}^n$. For example, let $\Omega$ be an open connected subset of $\mathbb{R}^n$. In the arguments used...
above, replace $R_m$ by $\{x \in \Omega: 1/m \leq \text{dist}(x, \partial \Omega) \leq 1/(m - 1)\}$, $S_m$ by $\{x \in \Omega: \text{dist}(x, \partial \Omega) = 1/m\}$ and $B_m$ by $\{x \in \Omega: 1/m \leq \text{dist}(x, \partial \Omega)\}$.

REFERENCES


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